Theory of signals and images I

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Image as a function

- Think of an **image** as a function, *f*,
- f: $R^2 \rightarrow R$
  - $I=f(x, y)$ gives the **intensity** at position (x, y)
  - The image only is defined over a rectangle, with a finite range:

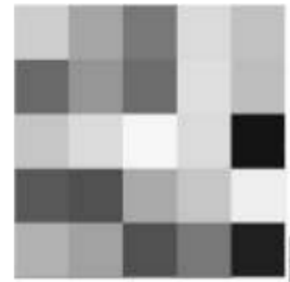
$$f: [a,b]\mathbf{x}[c,d] \rightarrow [0,1]$$

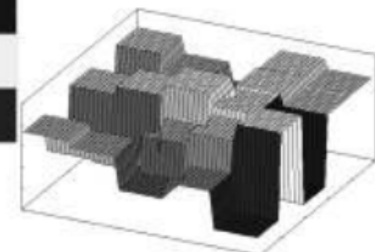
- A color image is just three dimensions:

$$I = f\,(x,\ y,\ c_{R,G,B})$$

Image

Image

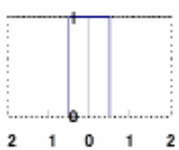




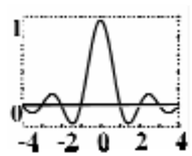
Modelo basado en píxeles
(constante por área cuadrada)

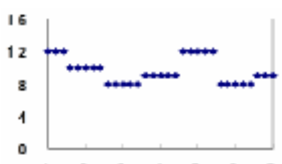
Discrete

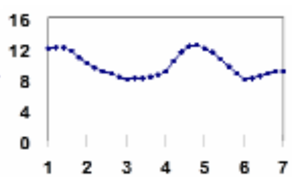
Continuity model



Funciones
base

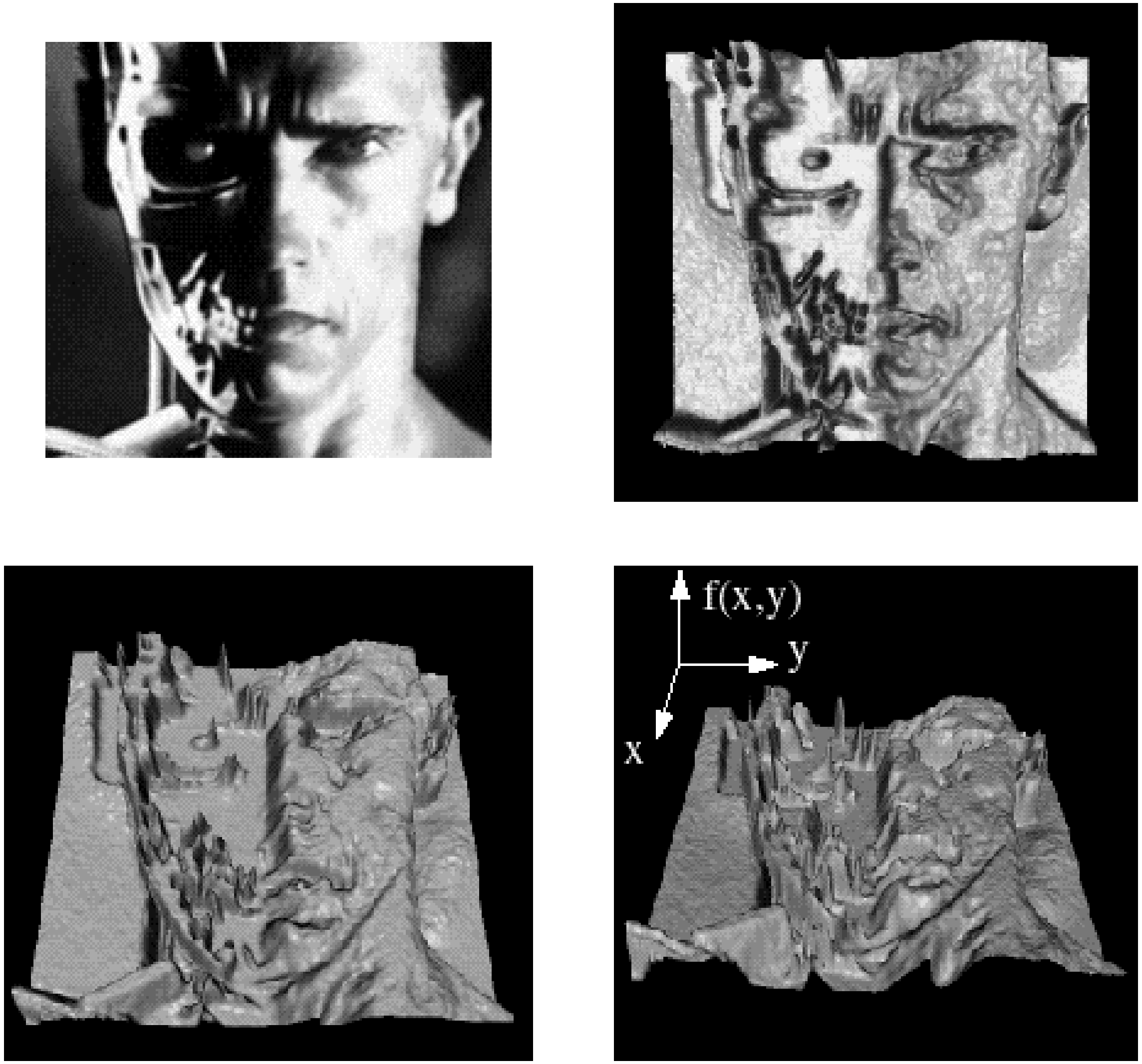






1D function

Image as a function



Linear Systems

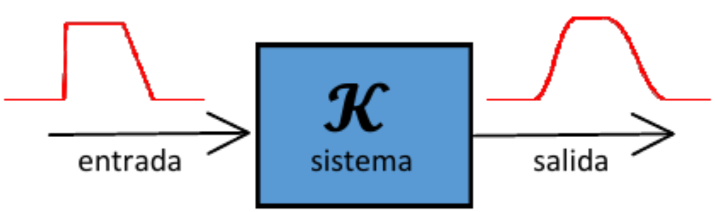
- Let define a new image *g* in terms of image *f*
  - We can transform either the domain or the range of *f*
- Range transformation:

$$g(x,y) = t(f(x,y))$$

- Preserve the range but change the domain of *f* :

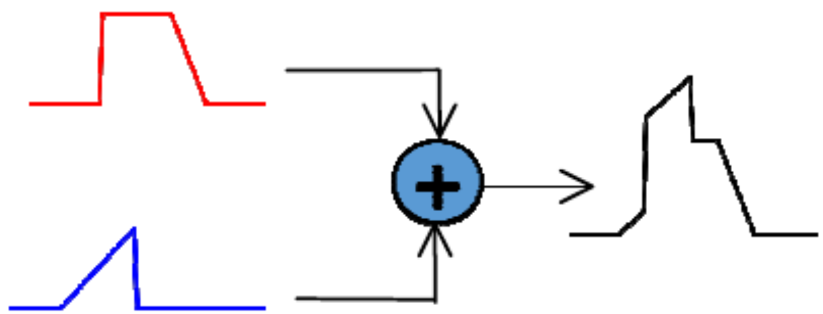
$$g(x,y) = f(t_x(x,y), t_y(x,y))$$

- other operations operate on both the domain and the range of *f*

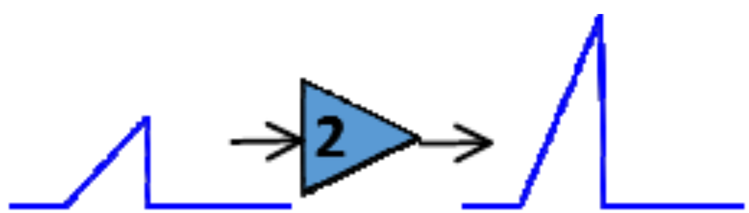


Linear Systems

- Sum: two function can be added:



- Scaling: a function can be multiplied by a constant



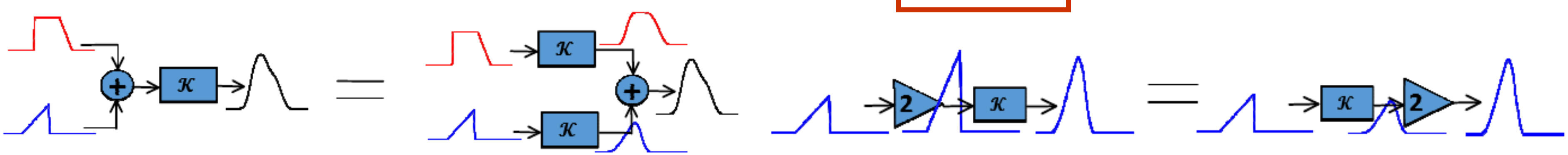
Linear Systems

Linear Shift Invariant Systems (LSIS)

- Linearity:

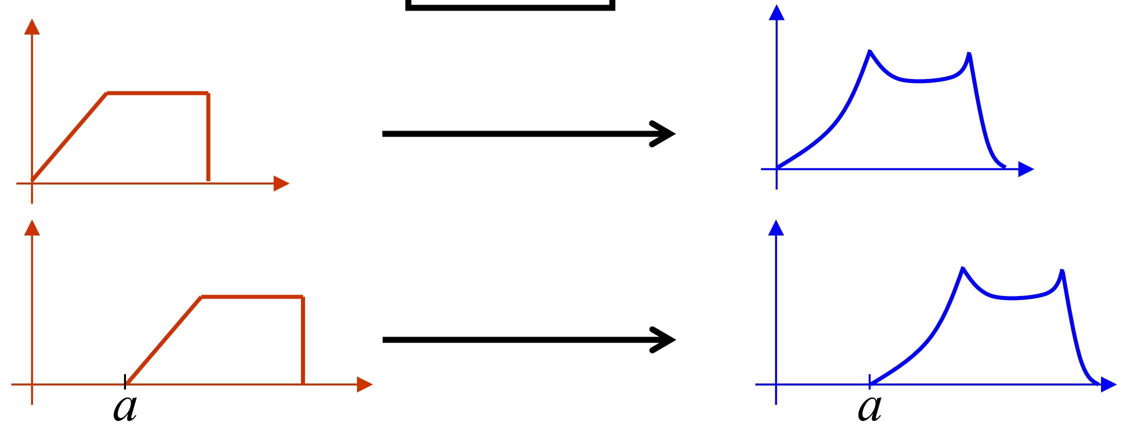
$$f_1 \rightarrow \boxed{} \rightarrow g_1 \qquad f_2 \rightarrow \boxed{} \rightarrow g_2$$

$$\alpha f_1 + \beta f_2 \rightarrow \boxed{} \rightarrow \alpha g_1 + \beta g_2$$



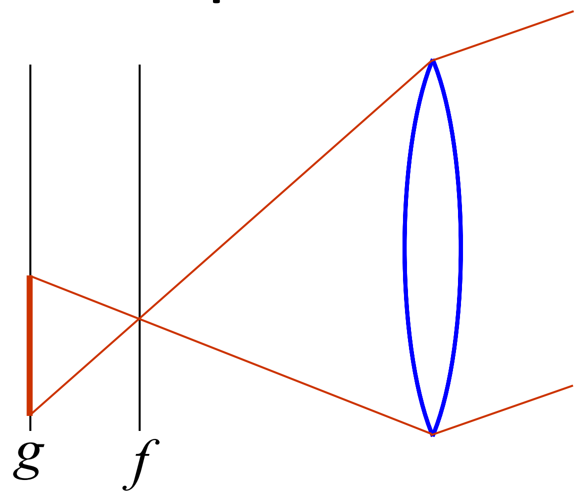
- Shift invariance:

$$f(x-a) \rightarrow \boxed{} \rightarrow g(x-a)$$



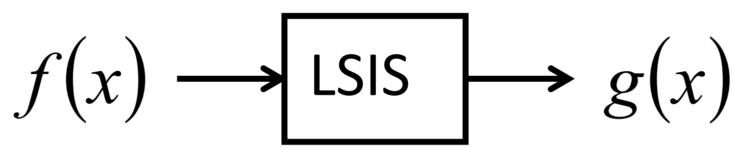
Linear Systems

- Example of LSIS



Defocused image (*g*) is a processed version of the focused image (*f*)

<u>Ideal lens is a LSIS</u>



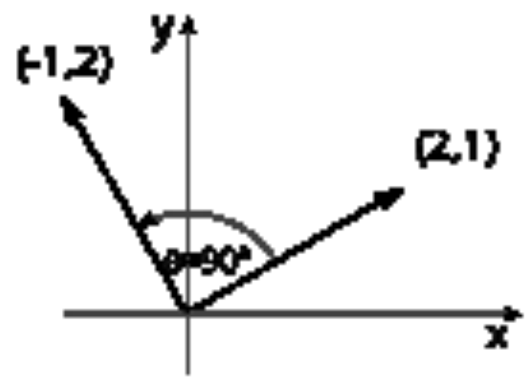
Linearity: Brightness variation

Shift invariance: Scene movement

(not valid for lenses with non-linear distortions)

Linear Systems

- Example (Rotation Matrix)
  - Let vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and R_θ the rotation matrix where rotated vector will be: $\begin{bmatrix} x_R \\ y_R \end{bmatrix} = g(x,y) = R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
  - Check if this function is linear
  - $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

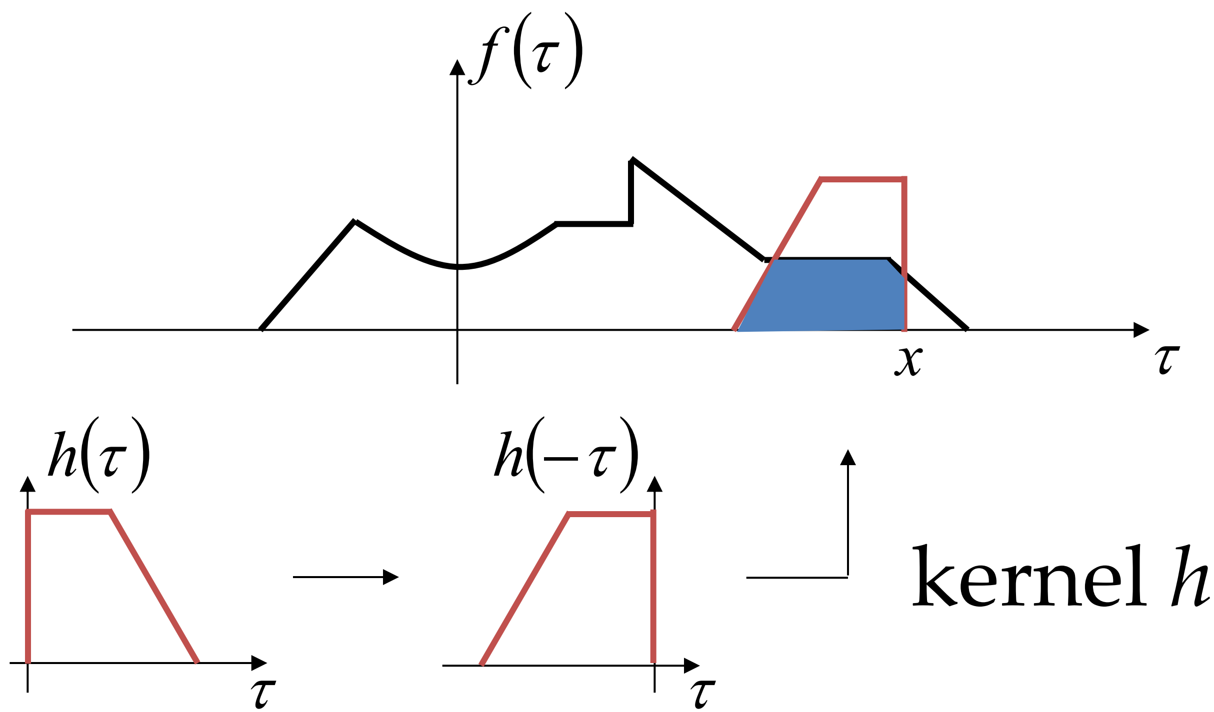


Linear Systems

- Convolution

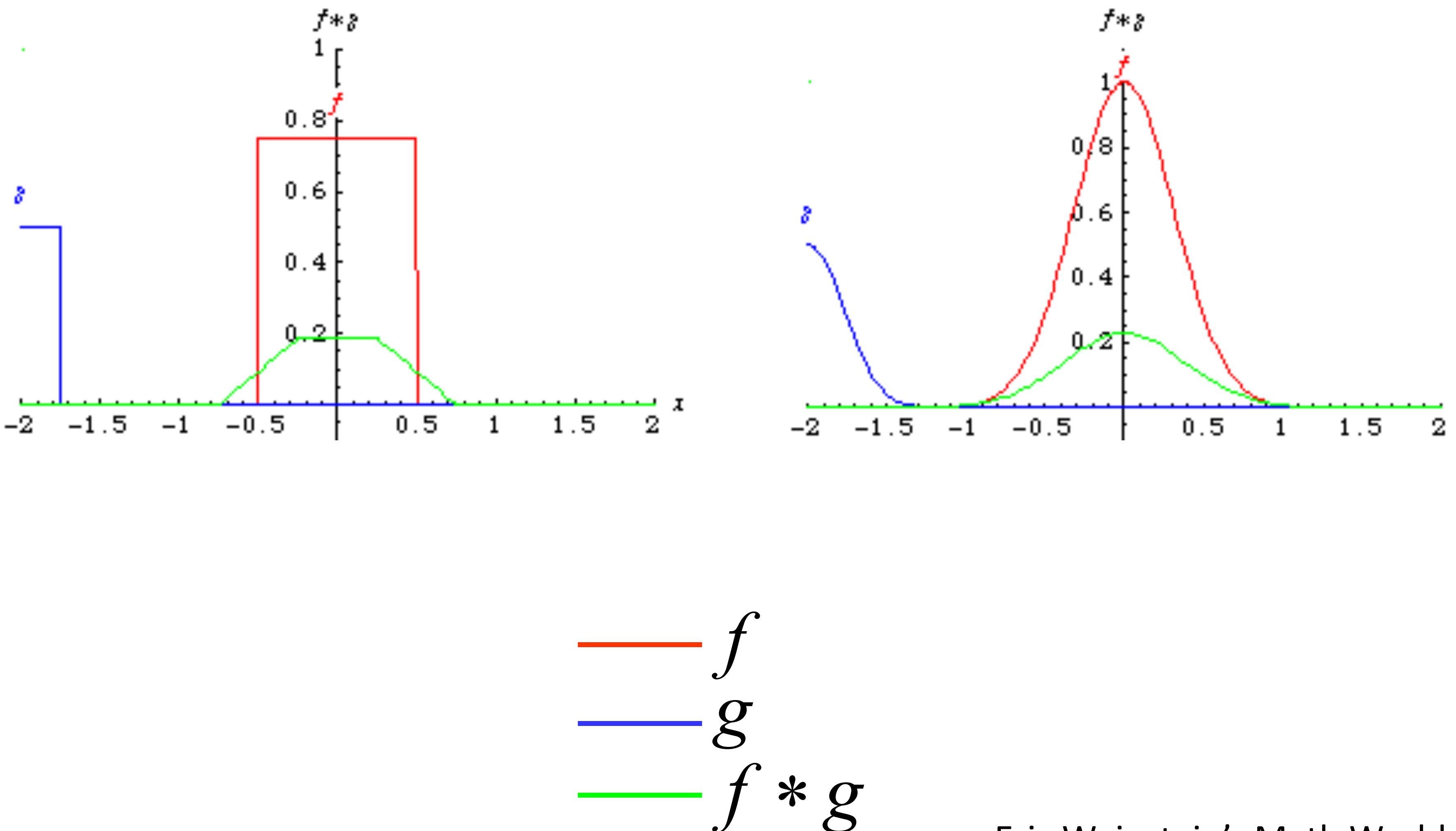
LSIS is doing convolution; convolution is linear and shift invariant

$$g(x) = \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau \qquad g = f * h$$



Linear Systems

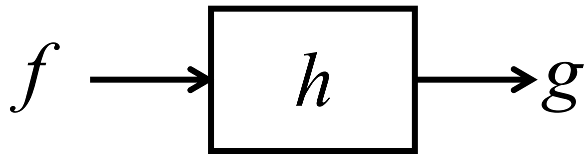
- Convolution



Eric Weinstein’s Math World

Linear Systems

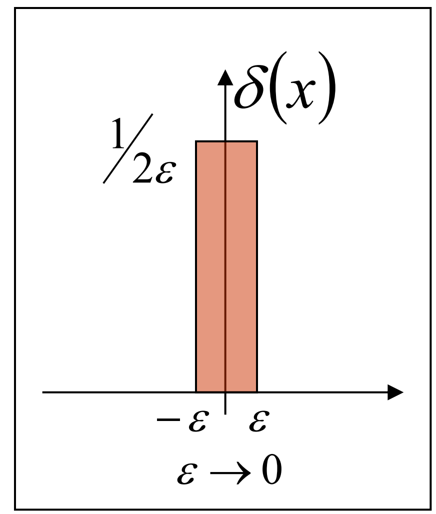
- Convolution



$$g = f * h$$

- What h will give us $g = f$?

Dirac Delta Function (Unit Impulse)



Shifting property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx$$

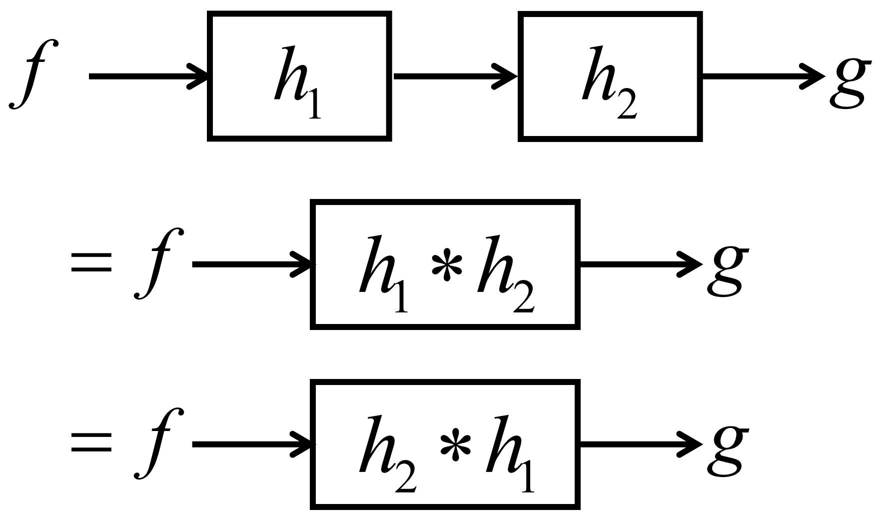
$$= f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$g(x) = \int_{-\infty}^{\infty} f(\tau)\delta(x-\tau)d\tau = f(x)$$

$$= \int_{-\infty}^{\infty} \delta(\tau)h(x-\tau)d\tau = h(x)$$

Linear Systems

- Convolution
- Commutative $$a * b = b * a$$
- Associative $$(a * b) * c = a * (b * c)$$
- Cascade system

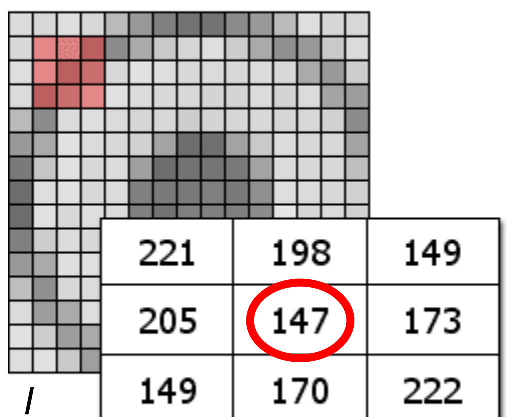


Linear Systems

- Convolution
  - A lot of filters are based on the concvolution http://en.wikipedia.org/wiki/Convolution
- Matrix convolution is an operation between two matrices.
  - image, *I*
  - *kernel*, *K*

-1	0	1
-2	0	2
-1	0	1

K



I

$$(K \otimes I)[x_i, j_i] = (-1 * 222) + (0 * 170) + (1 * 149) + (-2 * 173) + (0 * \mathbf{147}) + (2 * 205) + (-1 * 149) + (0 * 198) + (1 * 221) = 63$$

James Matthews, 2002
http://www.generation5.org/content/2002/convolution.asp

Image Filtering

- Signal to Noise Ratio (SNR)

Modeling: Noise is usually assumed to be additive and random

$$\hat{I}(x, y) = I(i, j) + n(i, j)$$

The observed intensity is the sum of the true intensity and a spurious and random signal.

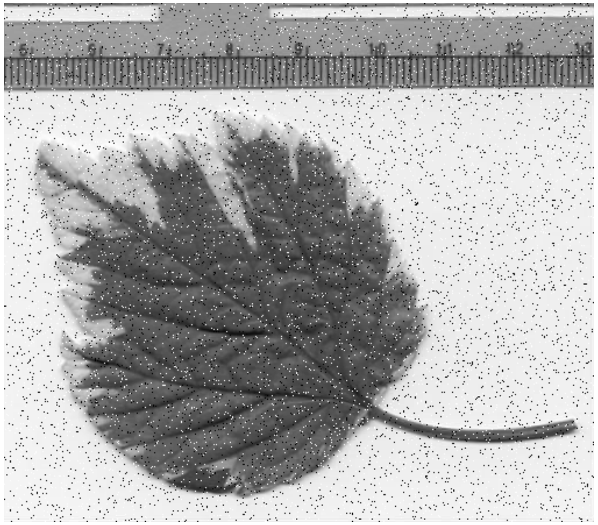
Signal-to-noise ratio, or SNR

$$SNR = \frac{\sigma_s}{\sigma_n}$$

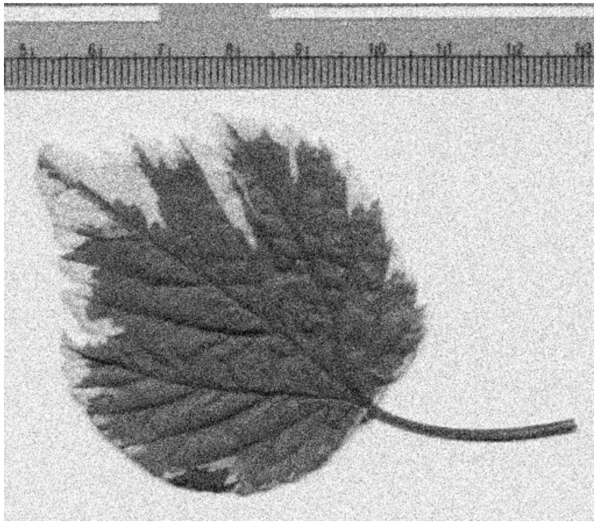
Ratio between std of signal and noise

Image Filtering

- Type of noises:
  - Salt and pepper
    - Spurious noise



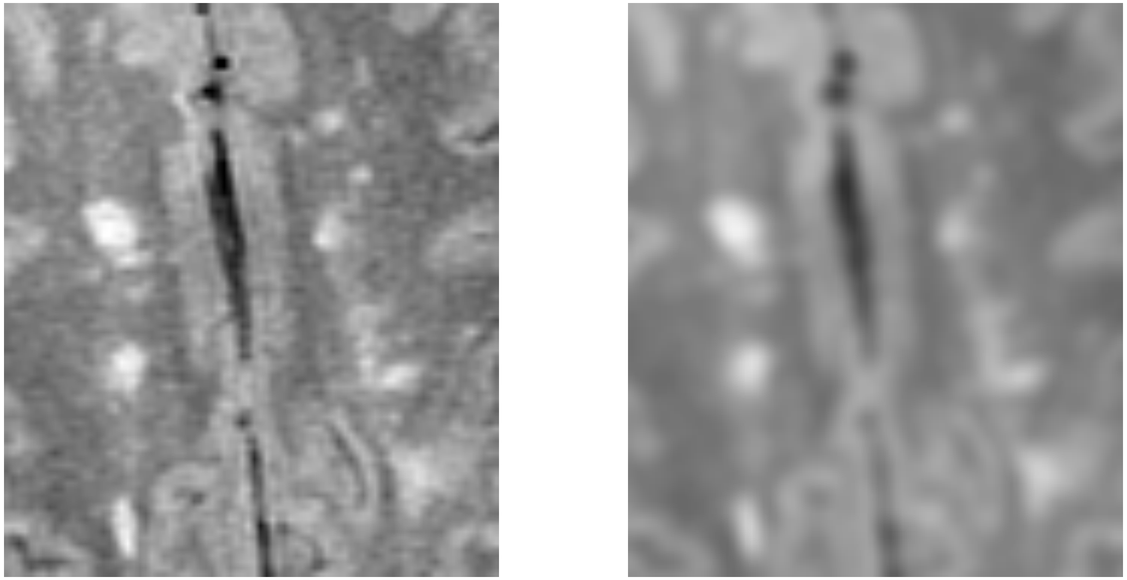
  - White noise
    - Normally zero mean Gaussian distribution



  - And others…

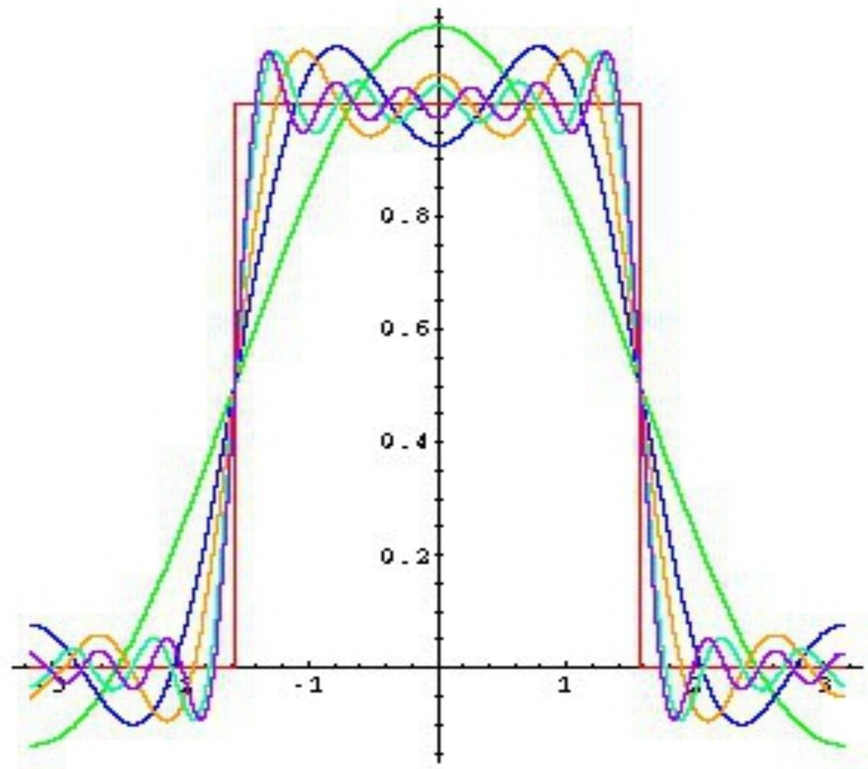
Image Filtering

- Objective
  - improve SNR
- Challenges
  - blurs object boundaries and smears out important structures



Serie de Fourier

A finales del siglo XVIII Jan Baptiste Joseph Fourier (1768-1830) descubrió un método que permite aproximar funciones periódicas mediante combinación lineal de funciones trigonométricas sencillas.



Serie de Fourier

Definición: Se llama serie de Fourier de una función f(x) en el intervalo [−L, L] a:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

Donde los coeficientes $\mathbf{a_0}$, $\mathbf{a_n}$ y $\mathbf{b_n}$ deben ser determinados.

Serie de Fourier

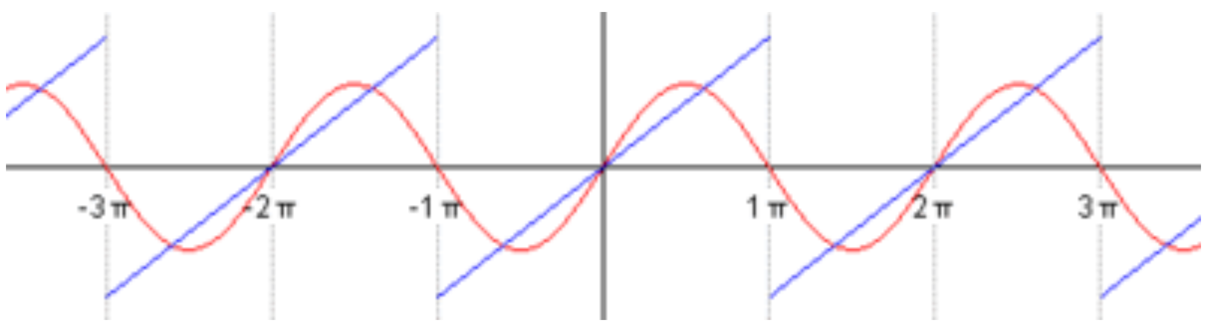
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

Los coeficientes $\mathbf{a_0}$, $\mathbf{a_n}$ y $\mathbf{b_n}$ están dados por:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \operatorname{sen} \frac{n\pi}{L} x \, \mathrm{d}x$$



Serie de Fourier

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x \qquad a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, \mathrm{d}x$$

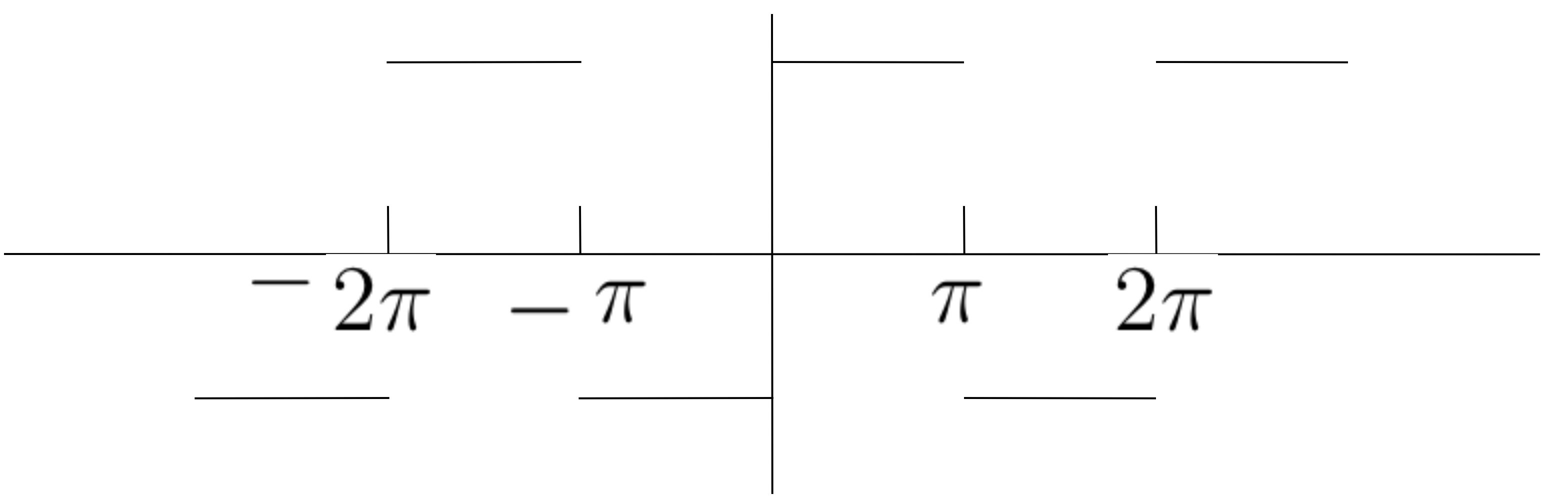
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \operatorname{sen} \frac{n\pi}{L} x \, \mathrm{d}x$$

$$\cos n\pi = (-1)^n \qquad \operatorname{sen} n\pi = 0$$

Serie de Fourier

Ejemplo: consideremos la función:

$$f(x) = \begin{cases} 1, & \text{si} \quad 0 \leq x \leq \pi; \\ -1, & \text{si} \quad \pi < x < 2\pi, \end{cases}$$



En este caso $2L = 2\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx$$

Laboratory for SCIAN Scientific Image Analysis

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f(x) = -1 entre -π y 0   f(x) = 1 entre 0 y π

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx$$

$$a_n = \frac{1}{\pi} [- \int_{-\pi}^{0} f(x) cos(nx) dx + \int_{0}^{\pi} f(x) cos(nx) dx]$$

$$a_n = \frac{1}{\pi}[-\int_{-\pi}^{0} f(x)cos(nx)dx + \int_{0}^{\pi} f(x)cos(nx)dx]$$

$$\int cos(nx)dx = \frac{1}{n}sen(nx)$$

evaluada en $0, \pi$ ó $-\pi$ es igual a 0, por lo tanto:

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sen(nx)dx$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[\frac{1}{n} - \frac{1}{n}cos(-n\pi) + \frac{1}{n} - \frac{1}{n}cos(n\pi)]$$

$$b_n = \frac{1}{\pi}[\frac{2}{n} - \frac{2}{n}cos(n\pi)]$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[\frac{2}{n} - \frac{2}{n}cos(n\pi)]$$

$$b_n = \frac{2}{n\pi}[1 - cos(n\pi)]$$

$$cos(n\pi) = +1, \quad n \quad par$$

$$cos(n\pi) = -1, \quad n \quad impar$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$b_n = \frac{2}{n\pi}[1 - cos(n\pi)]$$

$$cos(n\pi) = +1, \quad n \quad par$$

$$cos(n\pi) = -1, \quad n \quad impar$$

$$b_n = \frac{4}{n\pi}, \quad n \quad impar$$

$$a_0 = 0 \quad a_n = 0 \quad b_n = \frac{4}{n\pi}, \qquad n \quad impar \quad 2L = 2\pi$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

$$f(x) = \frac{4}{\pi} sen(x) + \frac{4}{3\pi} sen(3x) + \frac{4}{5\pi} sen(5x) + \cdots$$

$$f(x) = \frac{4}{\pi} [sen(x) + \frac{sen(3x)}{3} + \frac{sen(5x)}{5} + \cdots]$$

- Amplitud: $A_k = \sqrt{{a_k}^2 + {b_k}^2}$

- Fase: $\varphi_k = \tan^{-1}\frac{b_k}{a_k}$

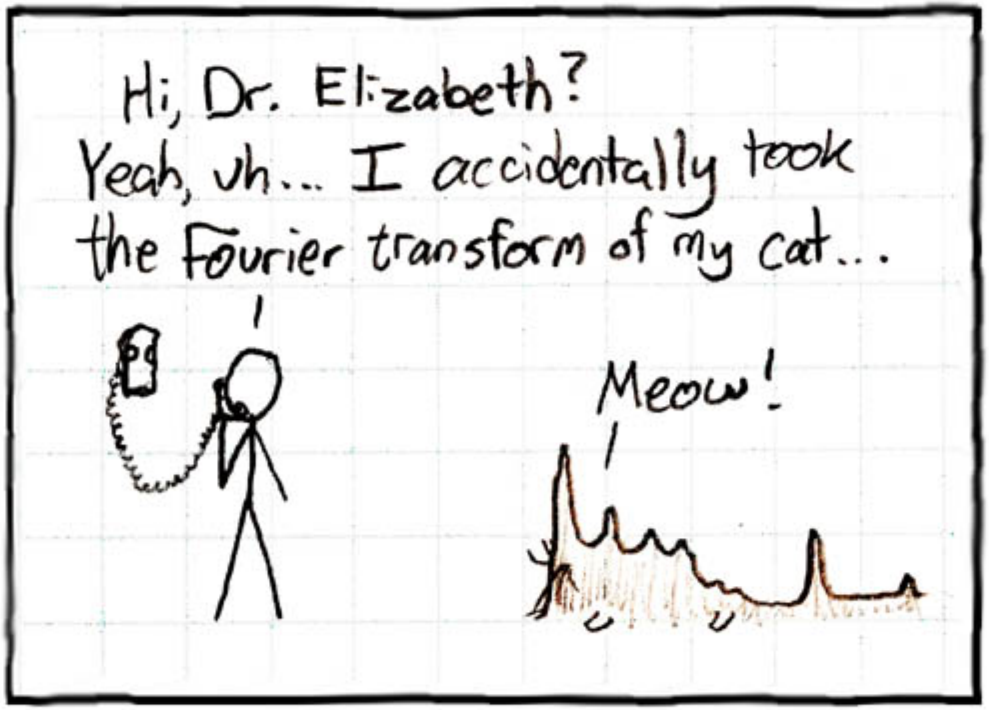
- Donde:

$$f(x) = \sum_{k \in \mathbb{Z}_+} a_k \cos(k\omega_0 x) + b_k \sin(k\omega_0 x)$$

Fourier Transform

- Frequency domain transformations (Fourier)

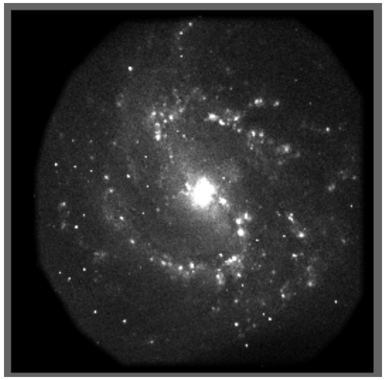
Jean-Baptiste Joseph Fourier (/ˈfʊəri.eɪ, -iər/;[1] French: [fuʁje]; 21 March 1768 – 16 May 1830) was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.[2]



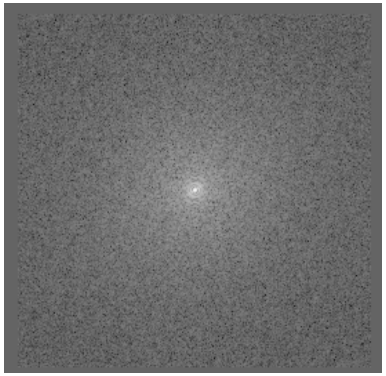
Joseph Fourier

Fourier Transform

Any periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies, called Fourier Series.



$$F(k,l) = \frac{1}{N^2} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} f(a,b)\, e^{-\iota 2\pi \left(\frac{ka}{N} + \frac{lb}{N}\right)}$$



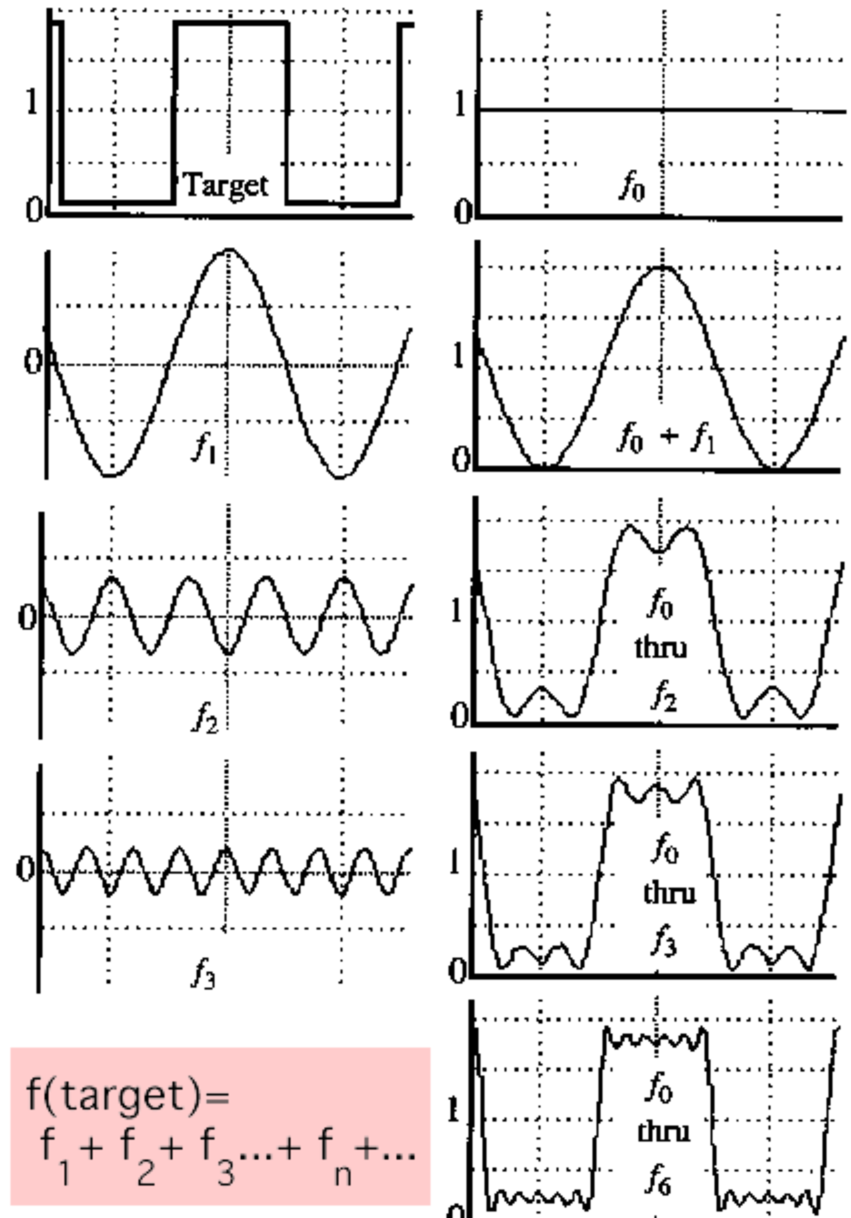
$$f(a,b) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k,l)\, e^{\iota 2\pi \left(\frac{ka}{N} + \frac{lb}{N}\right)}$$

Fourier Transform

- A sum of sinusoids
  - Building box

$$A\sin(\omega x + \phi)$$

- Assumptions
  - Periodic signals
  - More coefficients makes signal closer to the original.



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

Spatial domain **$f(x)$** → Fourier Transform → **$F(\omega)$** Frequency domain

- For every ω from 0 to inf, **$F(\omega)$** holds the amplitude *A* and phase ϕ of the corresponding sine

$$A \sin(\omega x + \phi)$$

  - How can *F* hold both? Complex number trick!

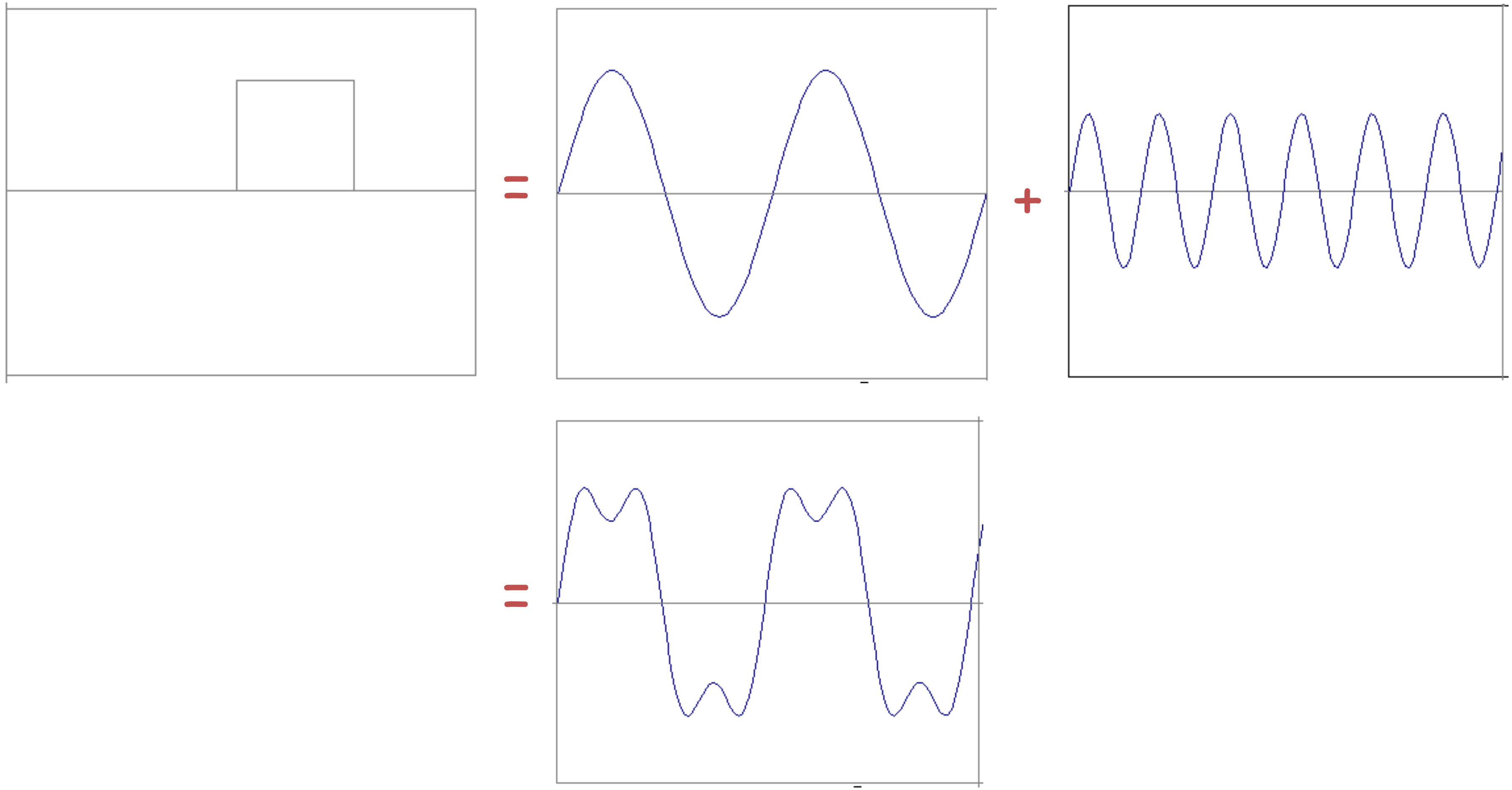
$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

$F(\omega)$ → Inverse Fourier Transform → **$f(x)$**

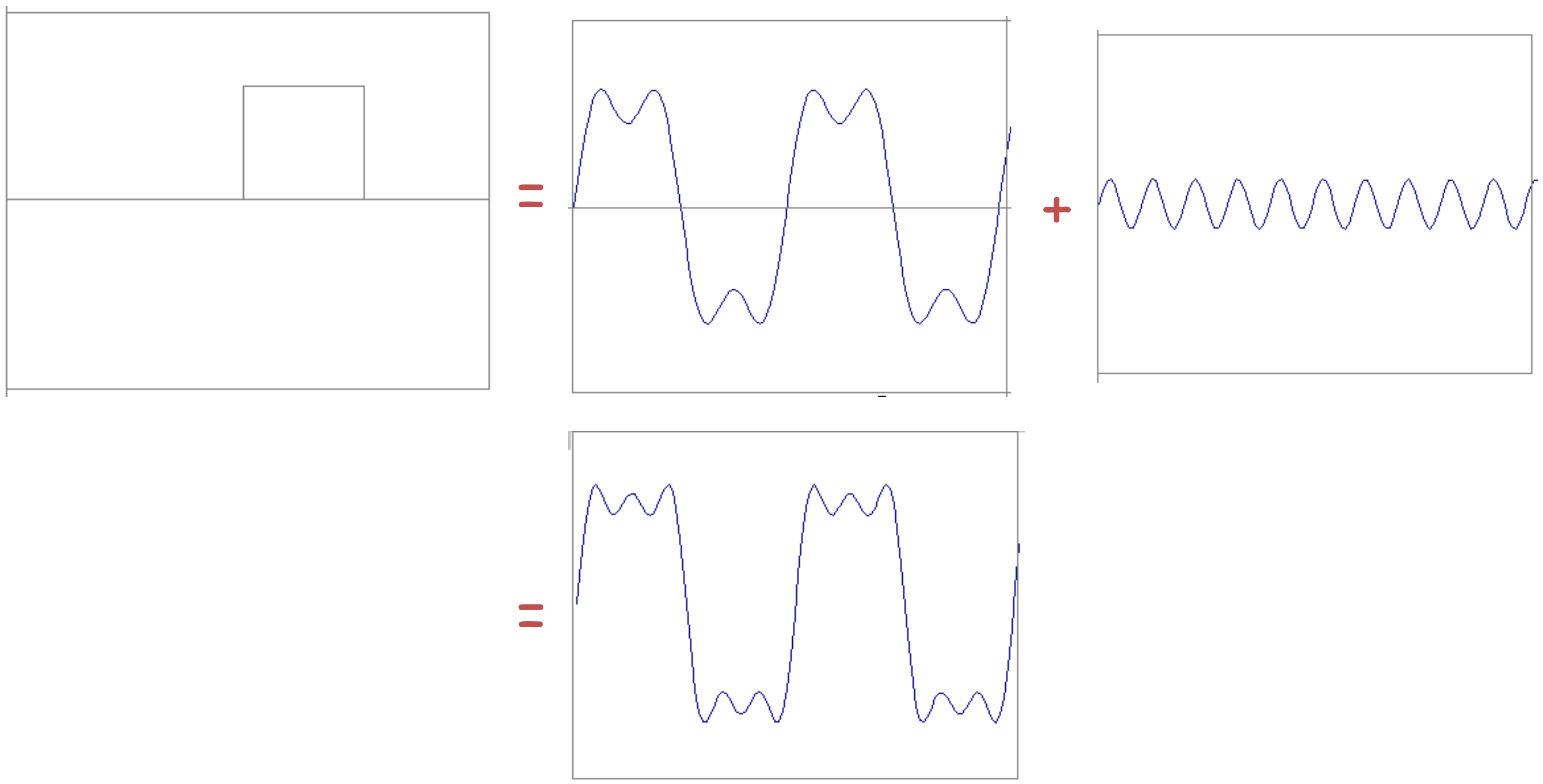
Fourier Transform

Example



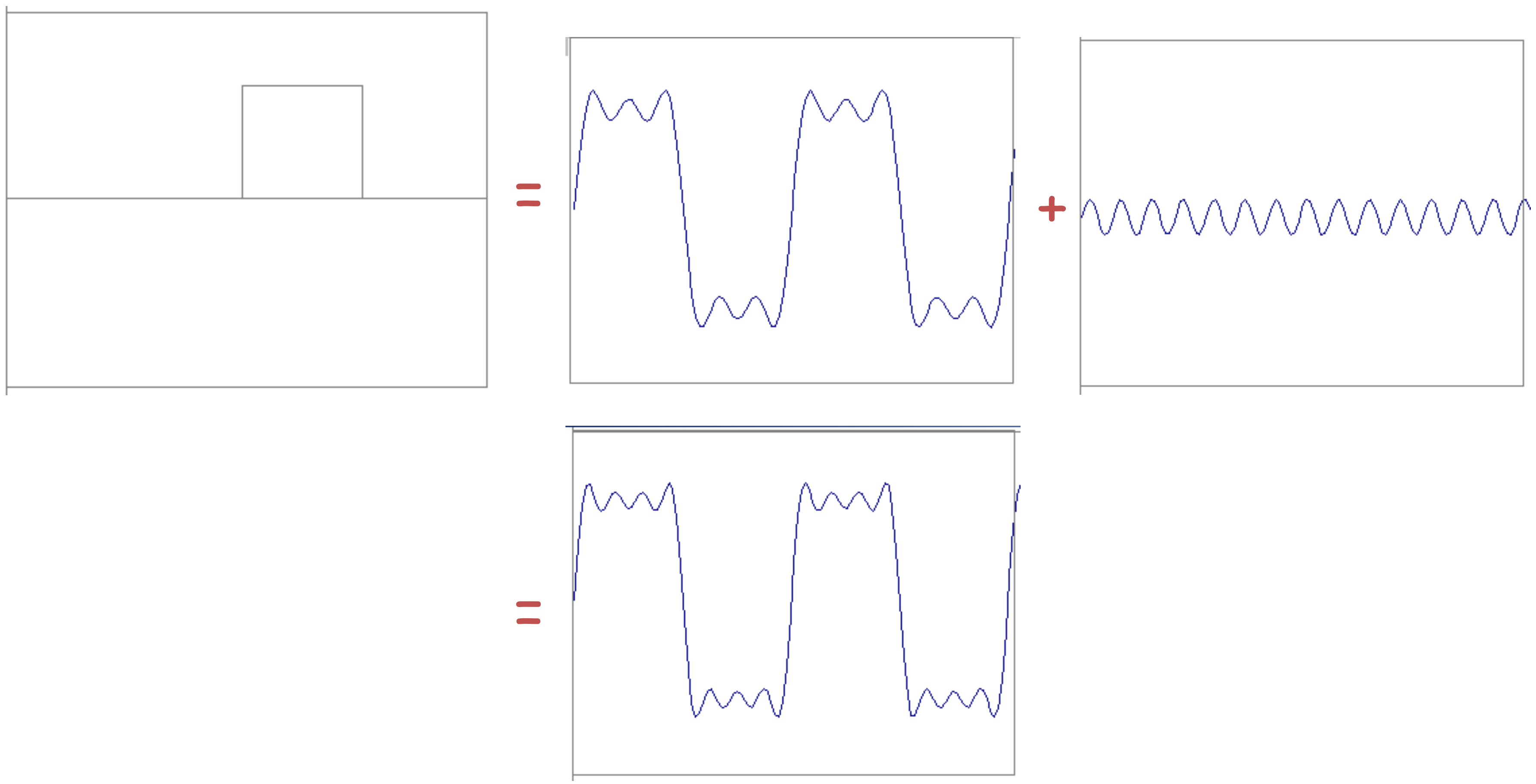
Fourier Transform

Example



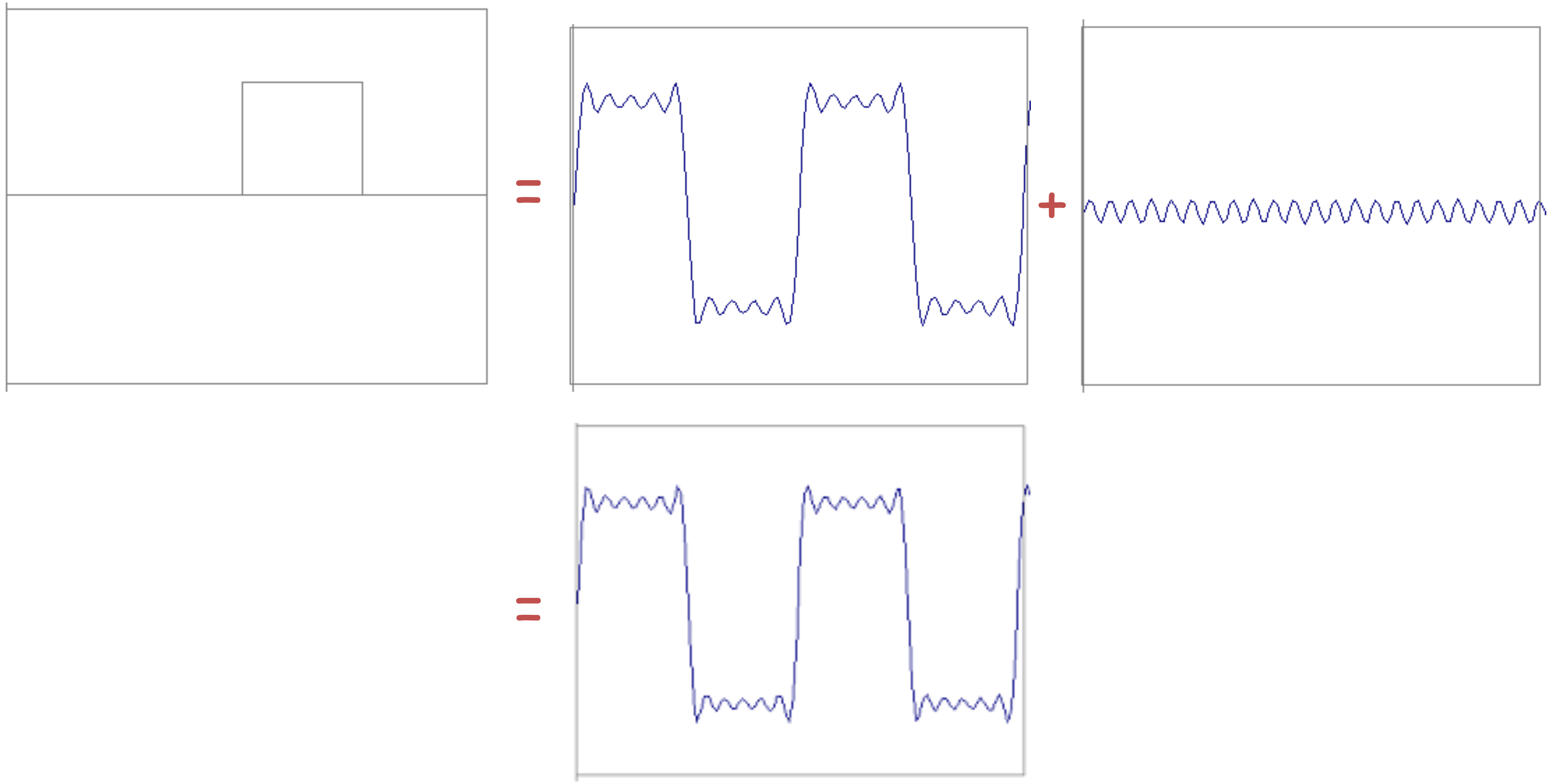
Fourier Transform

Example



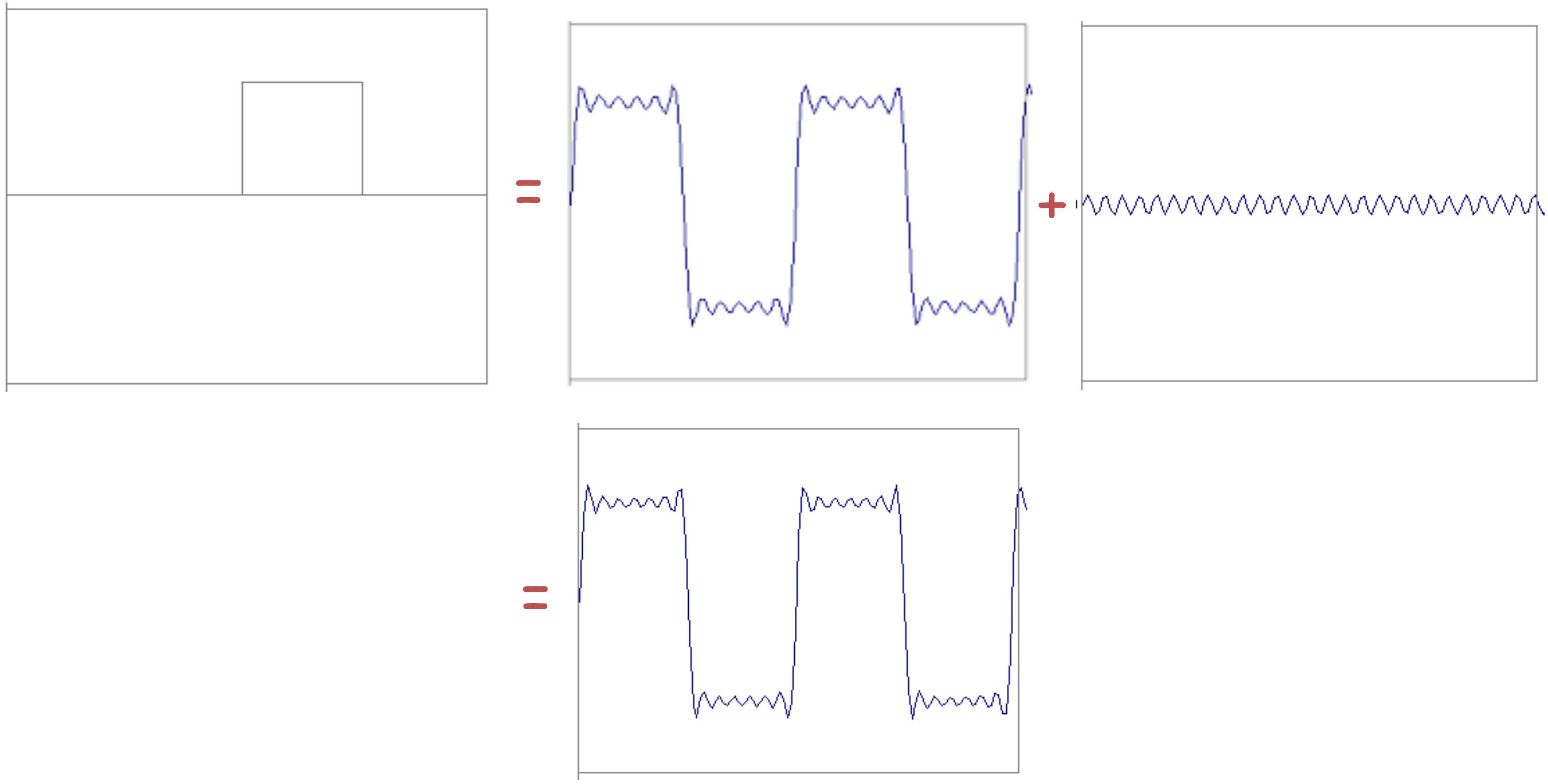
Fourier Transform

Example



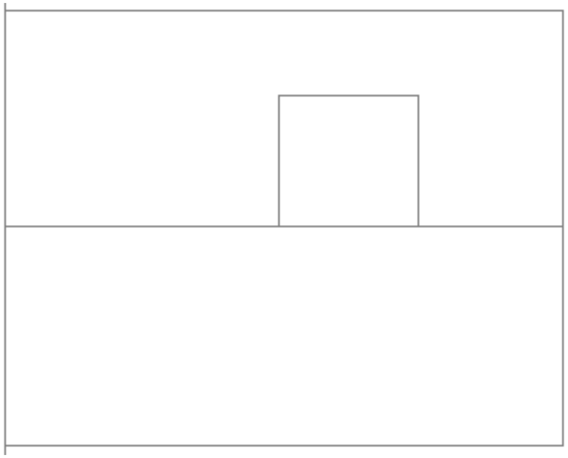
Fourier Transform

Example

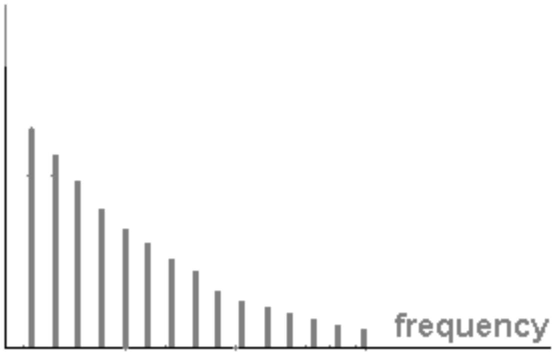


Fourier Transform

Example

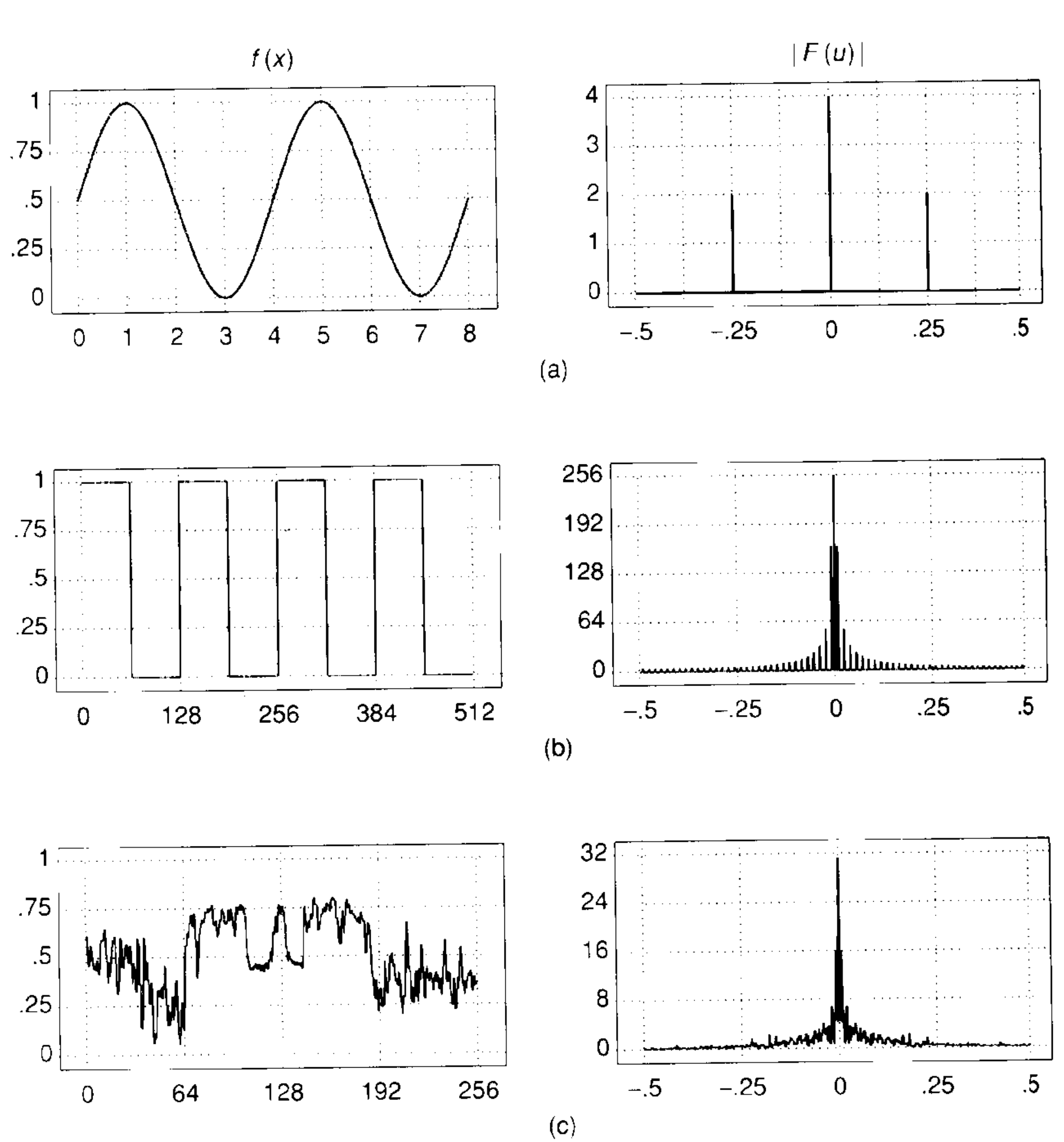


$$= \; A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$



Fourier Transform

Example

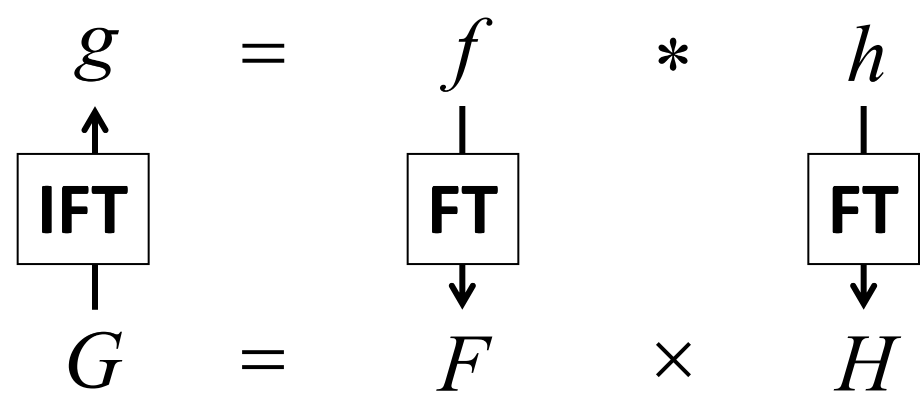


Fourier Transform

Fourier transform and convolution

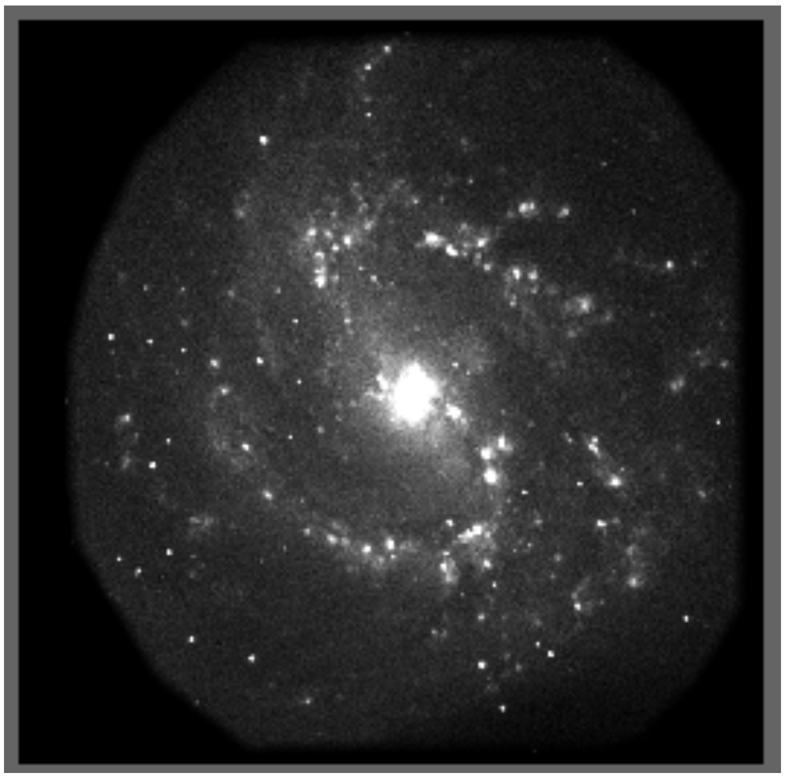
Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

So, we can find *g(x)* by Fourier transform

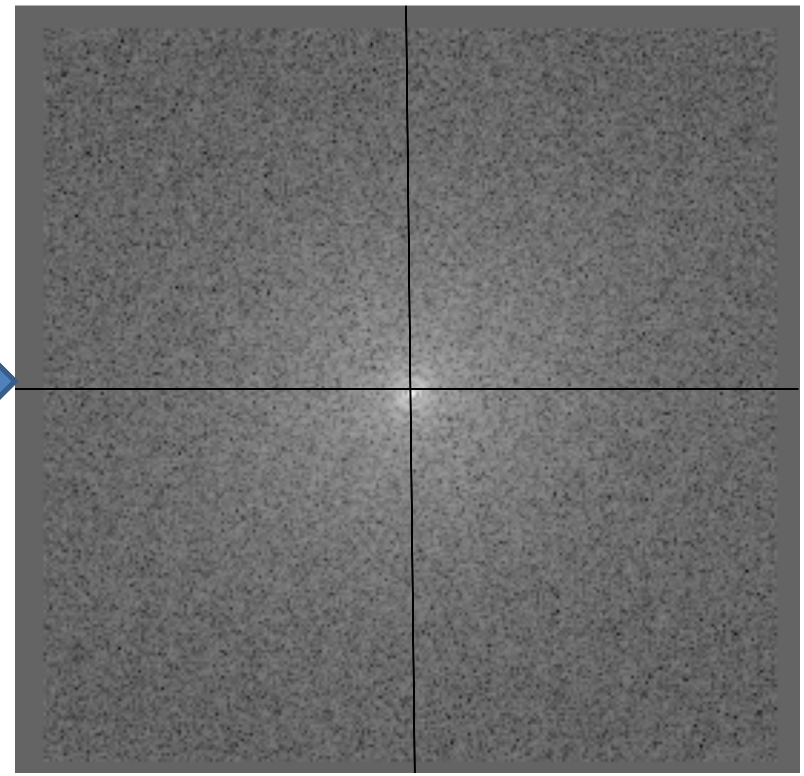


Fourier Transform

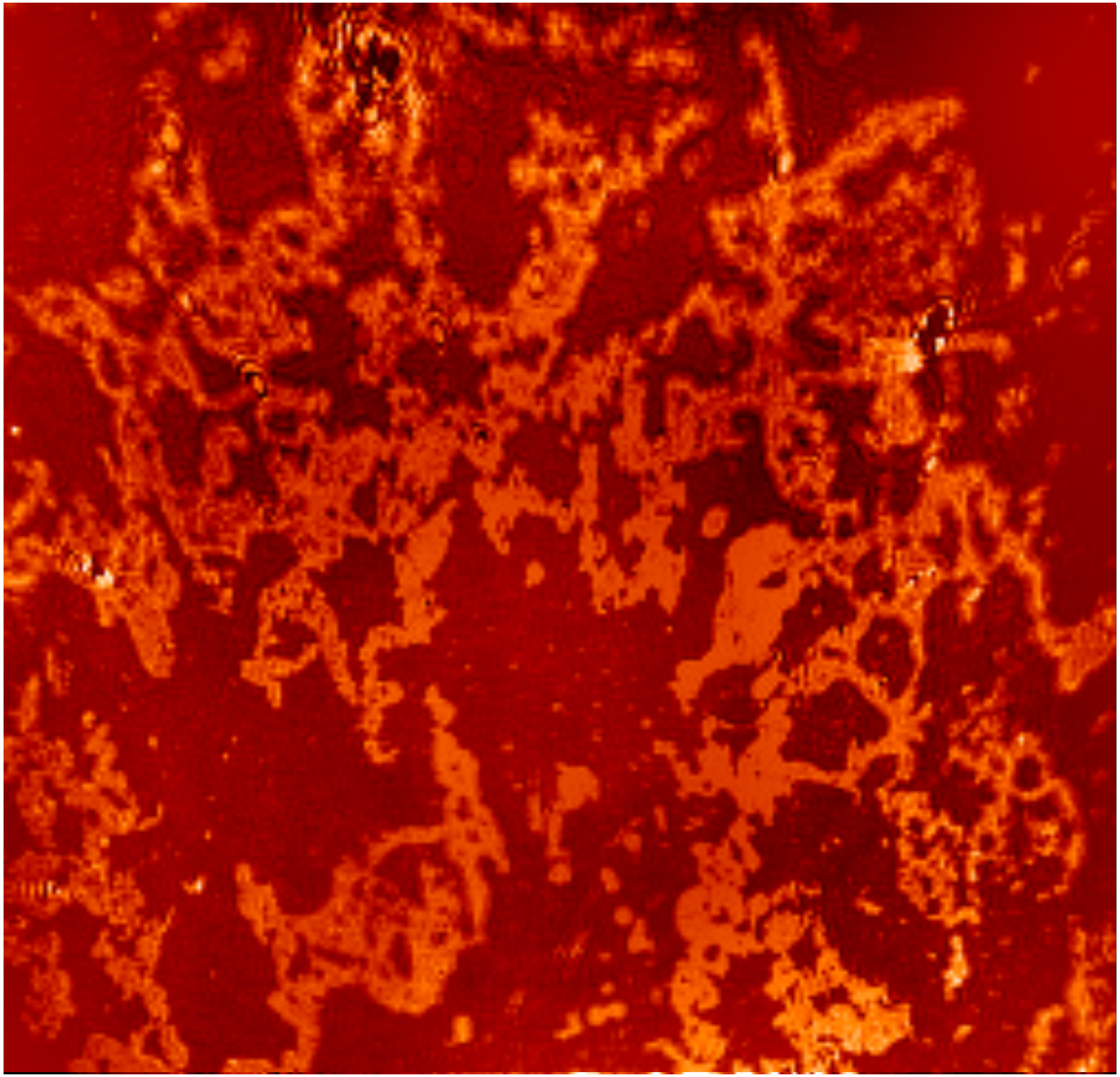
Fourier transform is symmetrical
x and y direction



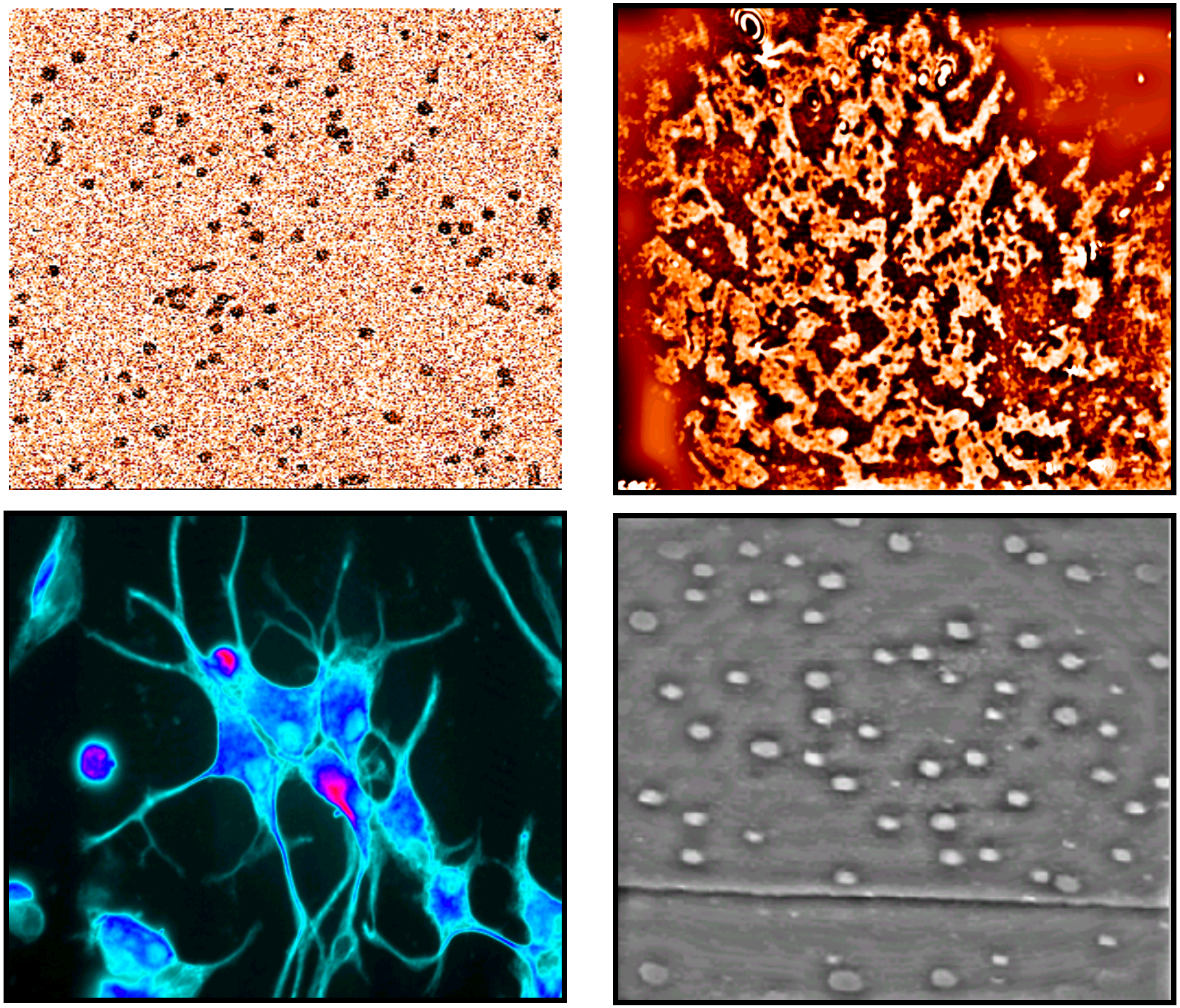
FFT



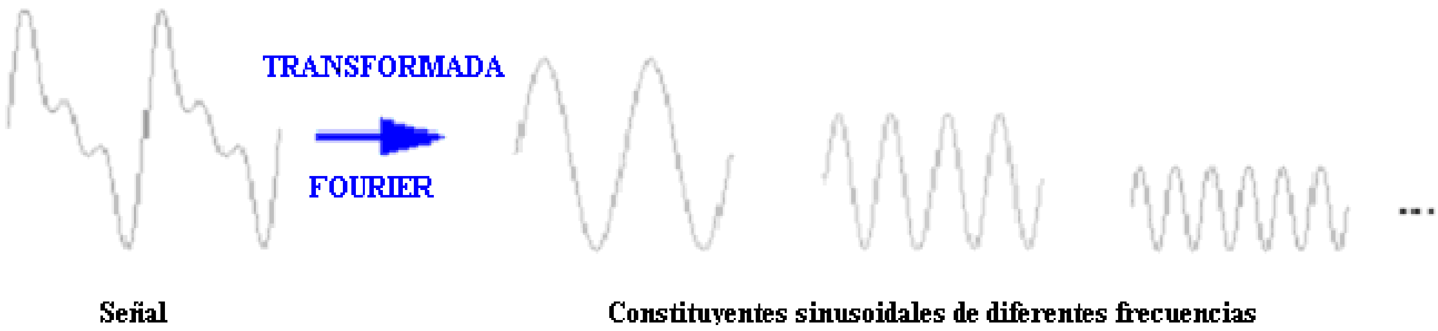
Fourier Transform

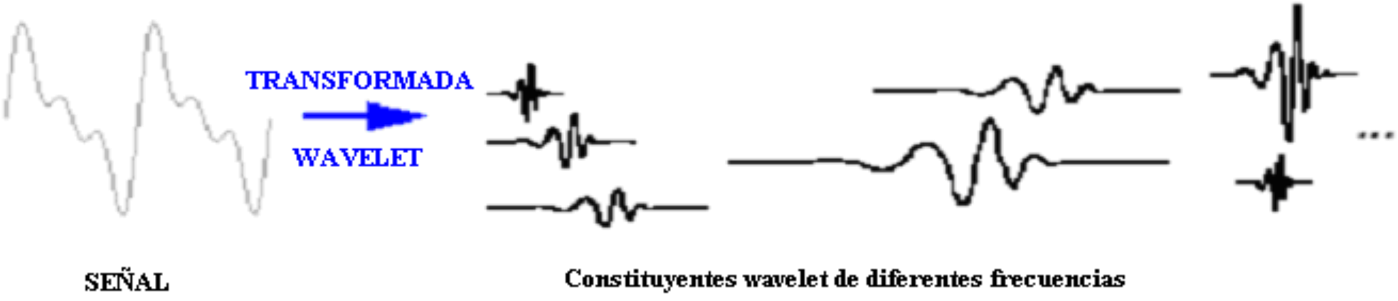


Fourier Transform



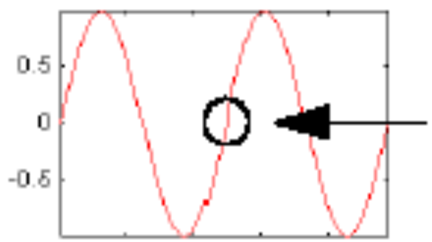
Wavelets Transform



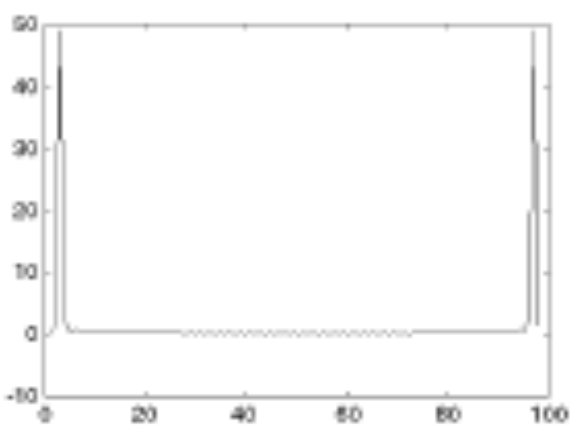


Wavelets Transform

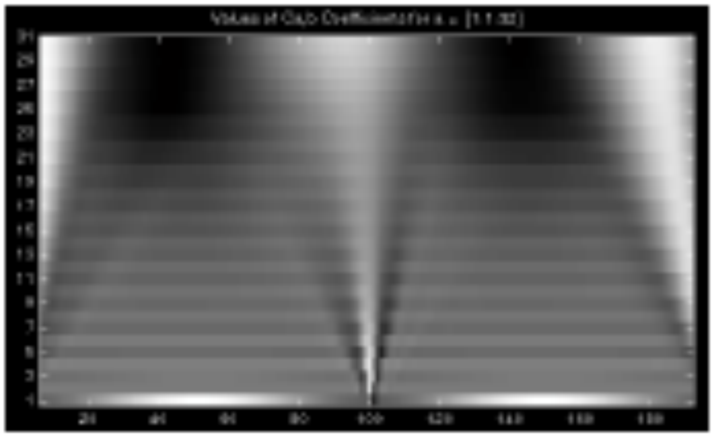
- Imagine a sinusoidal signal with a small discontinuity:



- **Fourier does not see the discontinuity.**
- **Wavelet shows exactly the location of the discontinuity in time.**



Fourier coeficients



Wavelet coeficients

Wavelets Transform

- Matematically, Fourier analysis representsed by the Fourier transform divide the original signal in a sum of sinusoidal signals.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt,$$

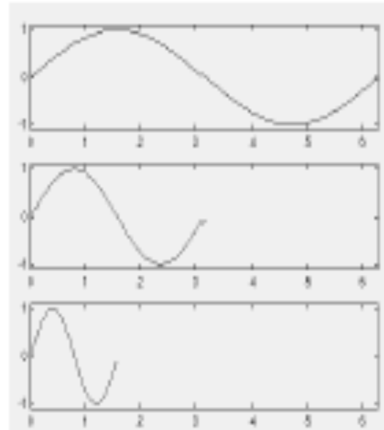
- **Wavelets transform is defined as a sum all time of the signal multiplied by a scale, changing the wavelet function. The wavelet coefficient result are then in terms of scale and position.**

$$C(scale, position) = \int_{-\infty}^{\infty} f(t) \psi(scale, position, t) dt$$

Wavelets Transform

- **Scaling of wavelet**

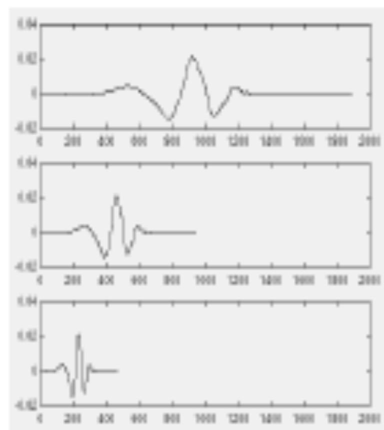
  - **Scale a wavelet means shrink or elongate is denominated scale factor**
  - **In sinusoidal the scale factor is easy to see….**



$f(t) = Seno(t)\ ;\ \ a = 1$

$f(t) = Seno(2t)\ ;\ \ a = 1/2$

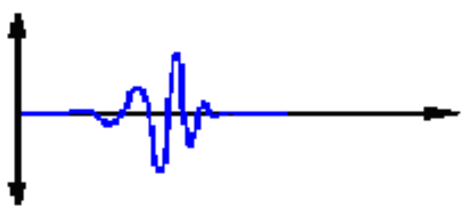
$f(t) = Seno(4t)\ ;\ \ a = 1/4$



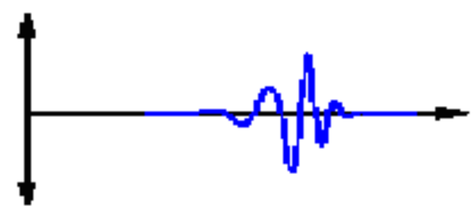
$f(t) = \Psi(t)\ ;\ \ a = 1$

$f(t) = \Psi(2t)\ ;\ \ a = 1/2$

$f(t) = \Psi(4t)\ ;\ \ a = 1/4$



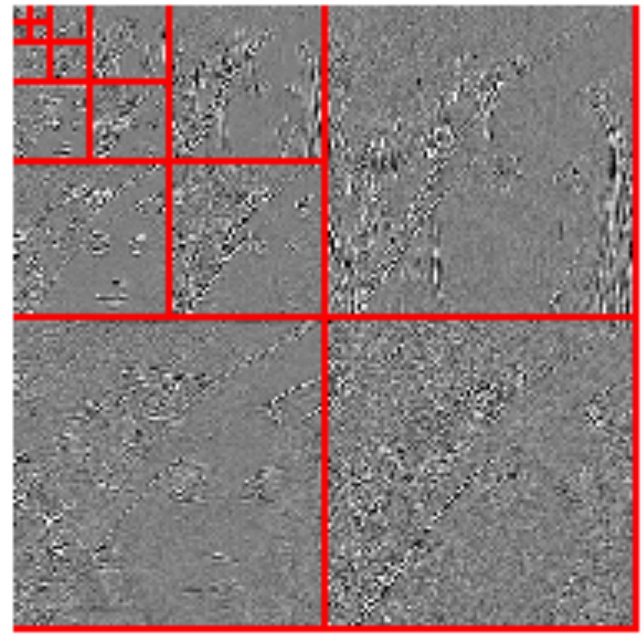
Función Wavelet $\Psi(t)$



Función Wavelet desplazada $\Psi(t - k)$

Image

Wavelet coefficients



Questions?