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FACULTAD DE MEDICINA
UNIVERSIDAD DE CHILE

Laboratory for SCIAN
Scientific Image Analysis

Theory of signals II

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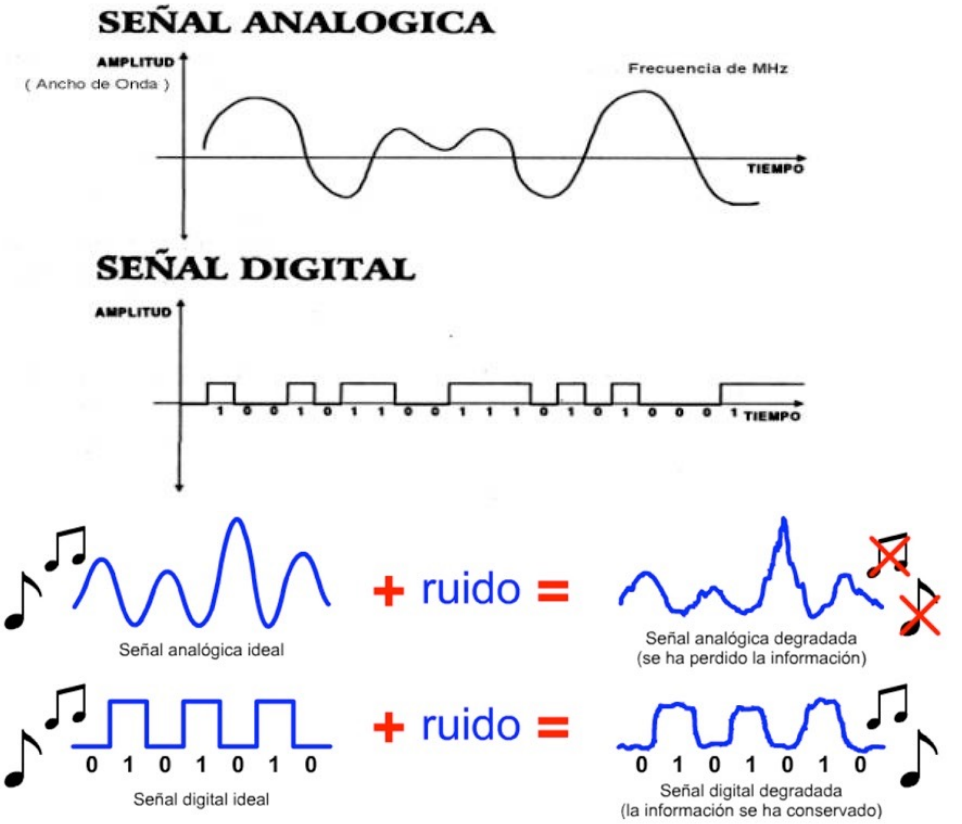
Analógo/Digital

- Señal Analógica:
  - Señal eléctrica que proviene de un sensor
  - Tiene ruido
  - Se atenúa
- Señal Digital:
  - Señal eléctrica codificada en 0 y 1
  - Se puede transmitir sin tener pérdidas de información
  - Se pierden las altas frecuencias
  - Muestreo 10 mayor que frecuencia máxima a representar (para mantener calidad)



¿Cómo lo entiende un computador?

- Computador sólo entiende números "bits" (valor 0 / 1)
- Conversión de señal analógica a digital
- Calidad depende de frecuencia de muestreo
- Se pueden almacenar/transmitir en "bits" (valor 0 / 1)

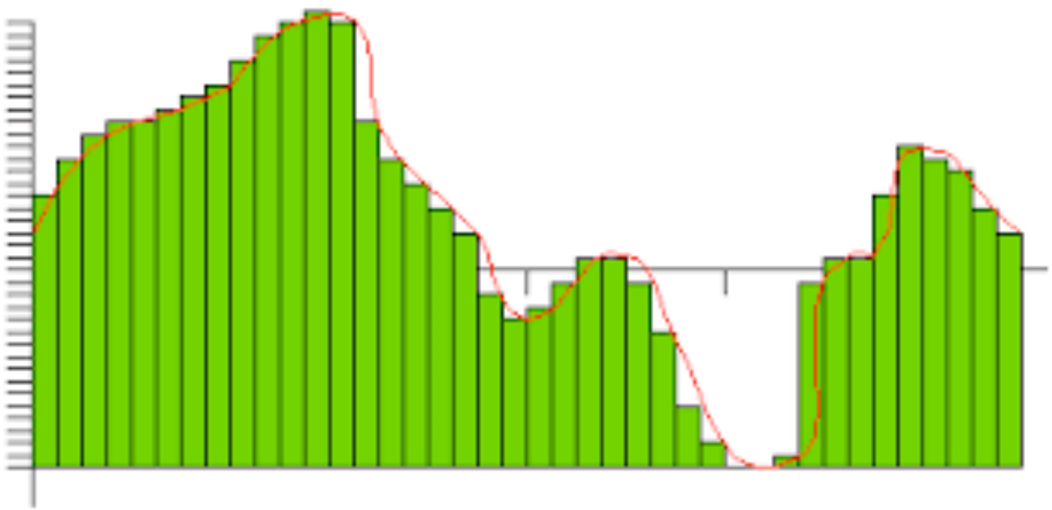


Image as a function

- Think of an **signal** as a function, *f*,
- f: R → R
  - $s=f(t)$ gives level at time (t)
  - The signal is defined in a certain time, with a finite range:

$$f: [t_0,t_1] \rightarrow [0,1]$$

- A multidimensional signal can be expressed as:

$$s = f(t, channel)$$

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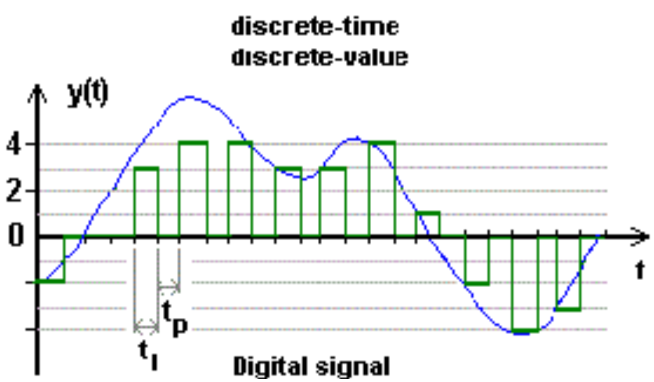
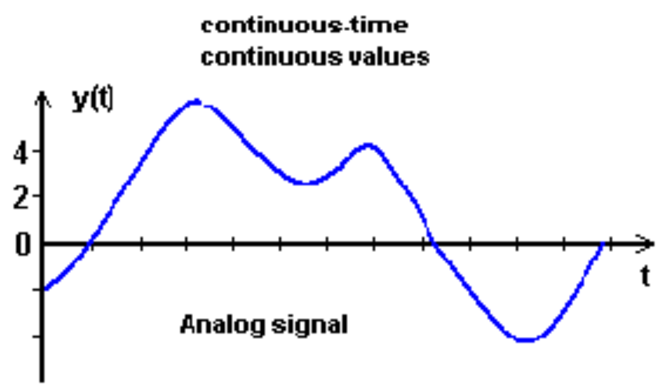
Signal

Discrete



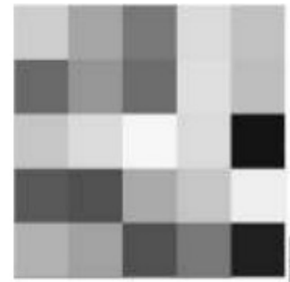
Continous



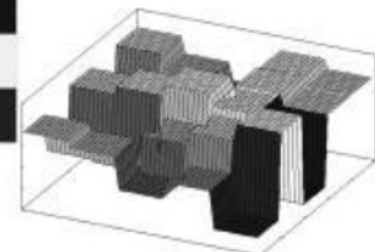
1D function

Image

Image

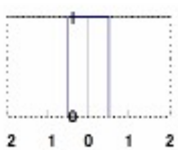


Modelo basado en píxeles
(constante por área cuadrada)

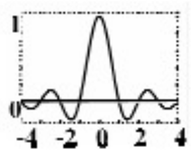


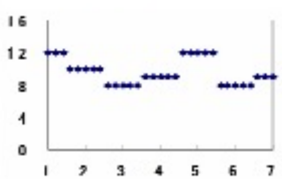
Discrete

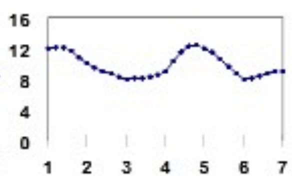
Continuity model



Funciones
base

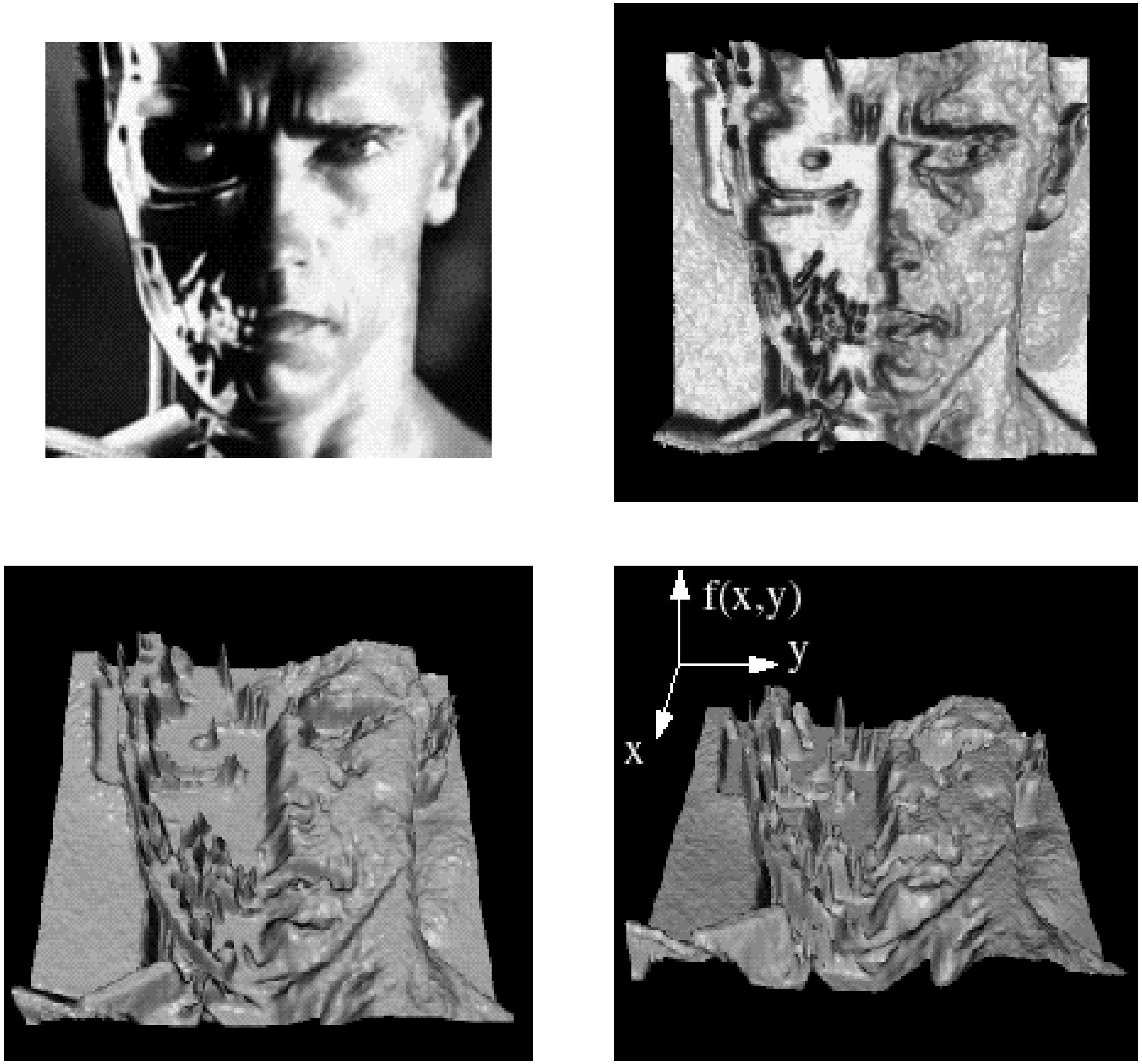






1D function

Image as a function



Linear Systems

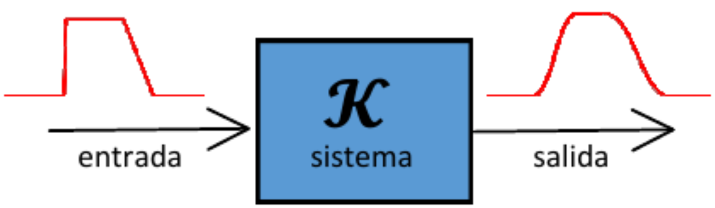
- Let define a new image *g* in terms of image *f*
  - We can transform either the domain or the range of *f*
- Range transformation:

$$g(x, y) = t(f(x, y))$$

- Preserve the range but change the domain of *f* :

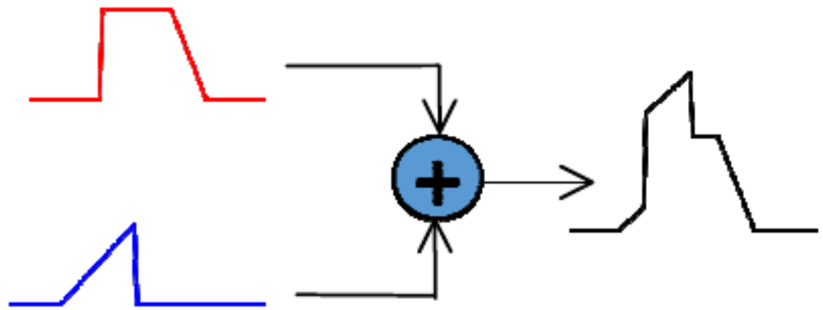
$$g(x, y) = f(t_x(x, y), t_y(x, y))$$

- other operations operate on both the domain and the range of *f*

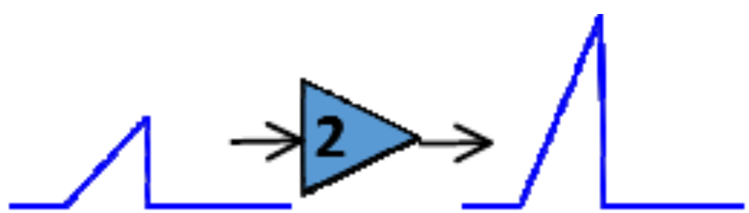


Linear Systems

- Sum: two function can be added:



- Scaling: a function can be multiplied by a constant



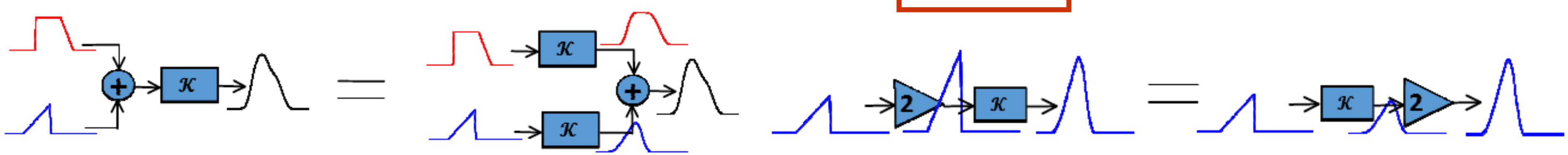
Linear Systems

Linear Shift Invariant Systems (LSIS)

$$f_1 \rightarrow \boxed{} \rightarrow g_1 \qquad f_2 \rightarrow \boxed{} \rightarrow g_2$$

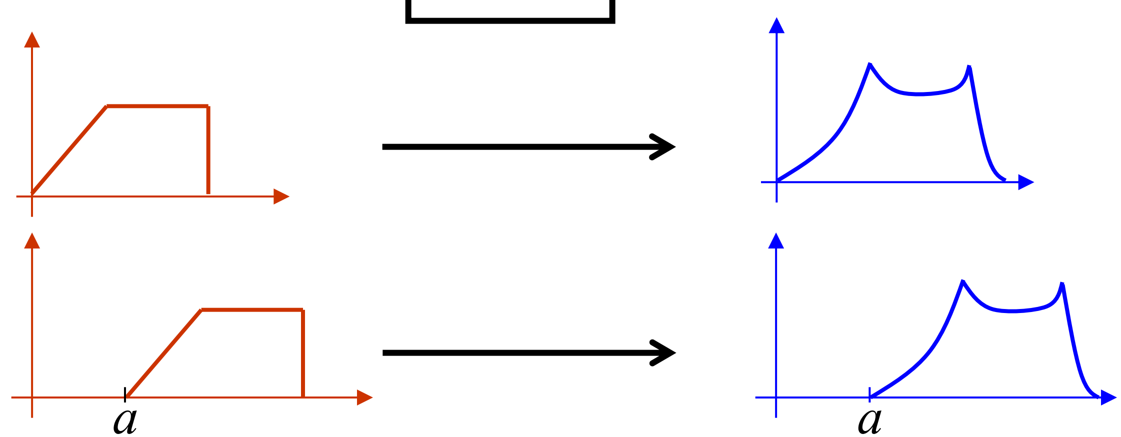
- Linearity:

$$\alpha f_1 + \beta f_2 \rightarrow \boxed{} \rightarrow \alpha g_1 + \beta g_2$$



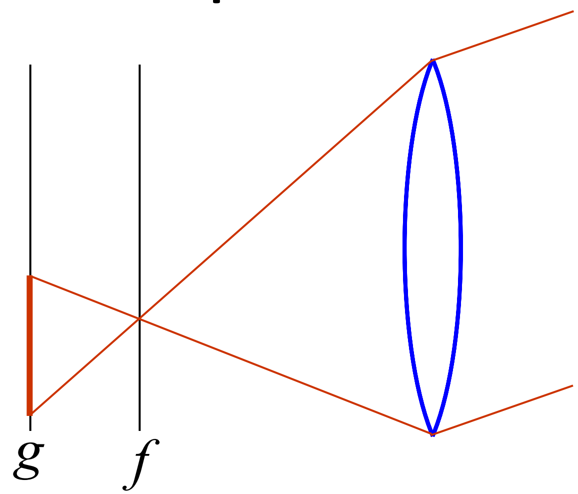
- Shift invariance:

$$f(x-a) \rightarrow \boxed{} \rightarrow g(x-a)$$



Linear Systems

- Example of LSIS



Defocused image (g) is a processed version of the focused image (f)

Ideal lens is a LSIS

$$f(x) \longrightarrow \boxed{\text{LSIS}} \longrightarrow g(x)$$

Linearity: Brightness variation

Shift invariance: Scene movement

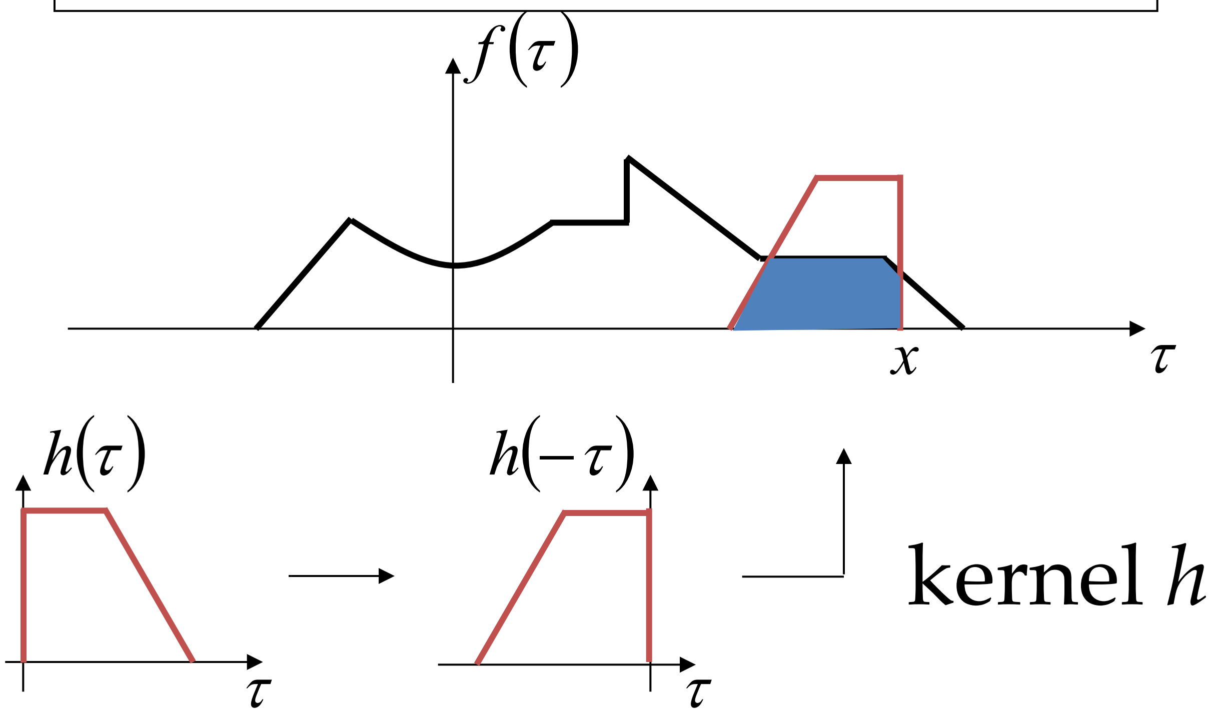
(not valid for lenses with non-linear distortions)

Linear Systems

- Convolution

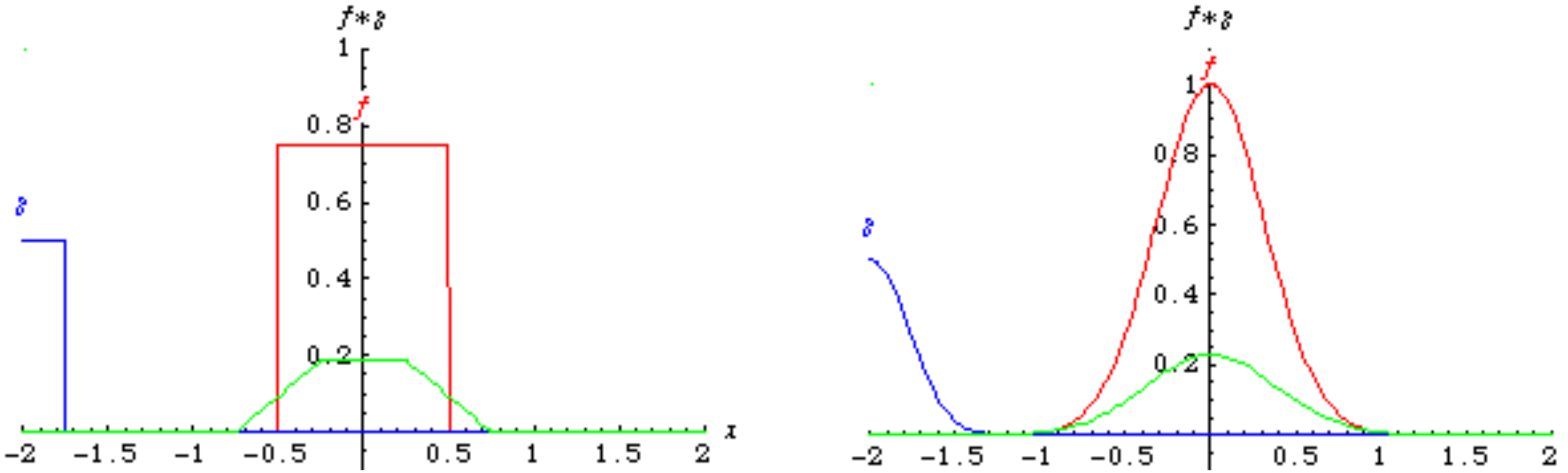
LSIS is doing convolution; convolution is linear and shift invariant

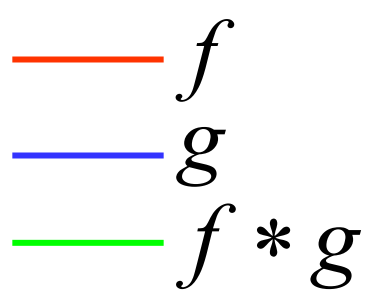
$$g(x) = \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau \qquad g = f * h$$



Linear Systems

- Convolution

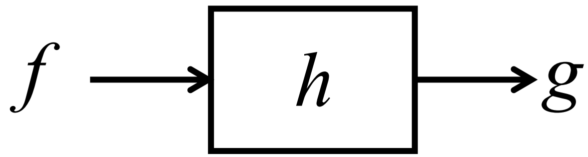




Eric Weinstein’s Math World

Linear Systems

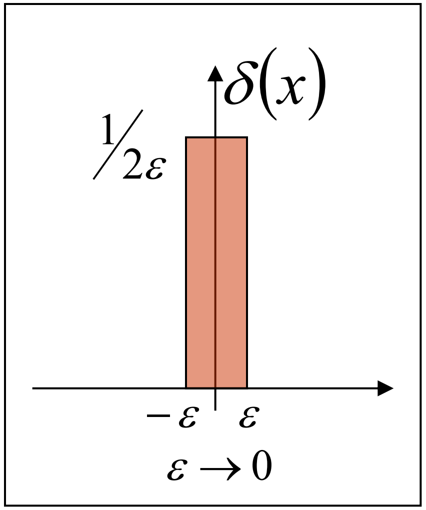
- Convolution



$$g = f * h$$

- What h will give us $g = f$?

Dirac Delta Function (Unit Impulse)



Shifting property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx$$

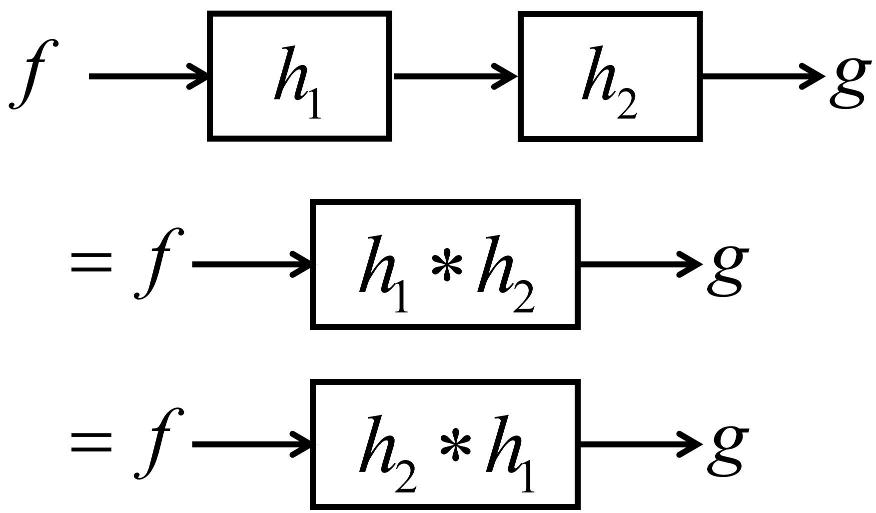
$$= f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$g(x) = \int_{-\infty}^{\infty} f(\tau)\delta(x-\tau)d\tau = f(x)$$

$$= \int_{-\infty}^{\infty} \delta(\tau)h(x-\tau)d\tau = h(x)$$

Linear Systems

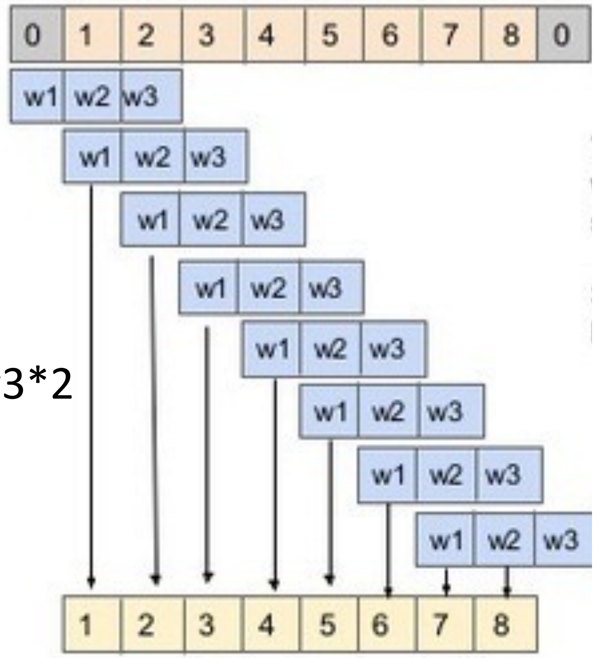
- Convolution
- Commutative $$a*b=b*a$$
- Associative $$(a*b)*c=a*(b*c)$$
- Cascade system



Linear Systems

- Convolution
  - A lot of filters are based on the convolution http://en.wikipedia.org/wiki/Convolution
- Convolution is an operation between two vectors.
  - signal, *s*
  - *kernel*, *K*

W1*0+w2*1+w3*2



James Matthews, 2002
http://www.generation5.org/content/2002/convolution.asp

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Linear Systems

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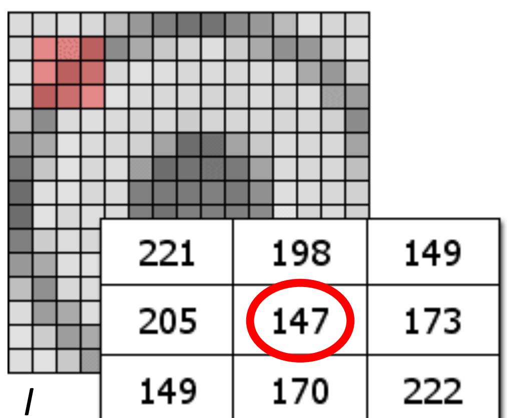
- Convolution
  - A lot of filters are based on the concvolution
    http://en.wikipedia.org/wiki/Convolution
- Matrix convolution is an operation between two matrices.
  - image, *I*
  - *kernel*, *K*

-1	0	1
-2	0	2
-1	0	1

K



I

$$(K \otimes I)[x_i, j_i] = (-1 * 222) + (0 * 170) + (1 * 149) + (-2 * 173) + (0 * \mathbf{147}) + (2 * 205) + (-1 * 149) + (0 * 198) + (1 * 221) = 63$$

James Matthews, 2002
http://www.generation5.org/content/2002/convolution.asp

Filtering

- Signal to Noise Ratio (SNR)

Modeling: Noise is usually assumed to be additive and random

$$\mathrm{S(t)} = s(t) + n(t)$$

The observed signal is the sum of the true signal and a spurious and random signal.

Signal-to-noise ratio, or SNR

$$SNR = \frac{\bar{S}}{\sigma_n}$$

Ratio between average of signal and std of noise

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Filtering

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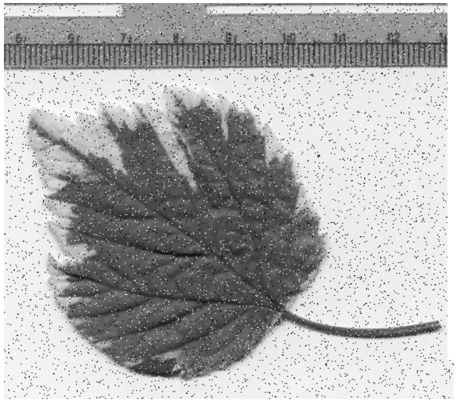
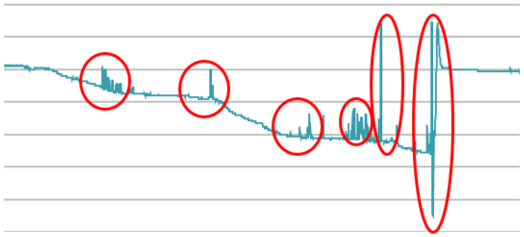
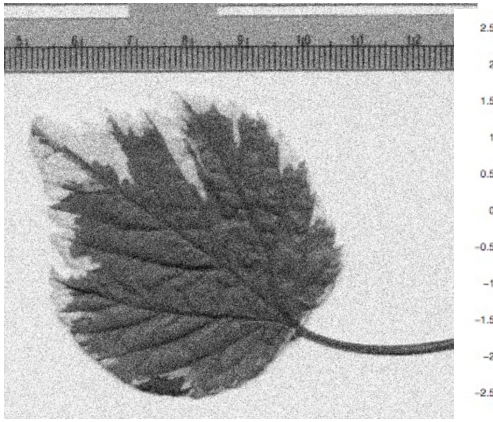
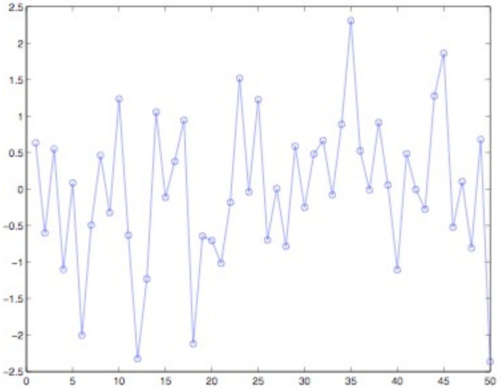
- Type of noises:
  - Salt and pepper
    - Spurious noise





  - White noise
    - Normally zero mean Gaussian distribution

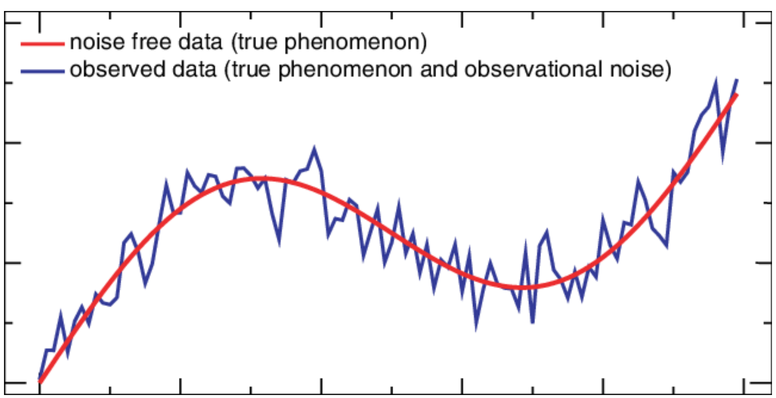


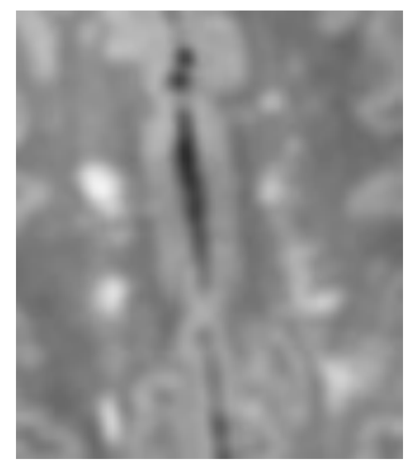


  - And others…

Filtering

- Objective
  - improve SNR
- Challenges
  - blurs original signal and smears out important patterns





Filtering

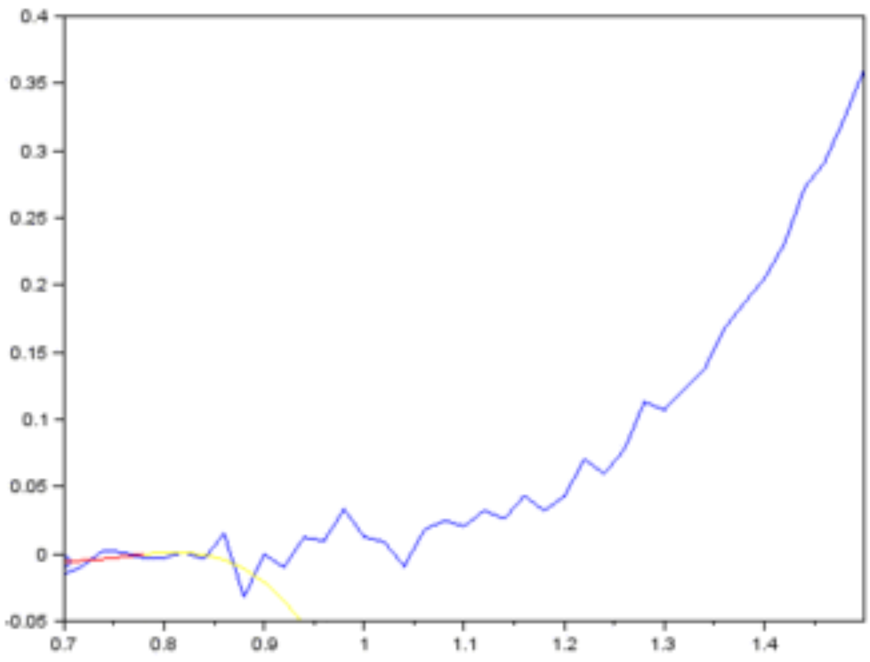
- Define Window
  - To filter we have to define size windows
    - It depends on noise frequencies
    - Signal frequency
    - Sampling frequency
  - NOTE: It can erase high frequencies!

Filtering

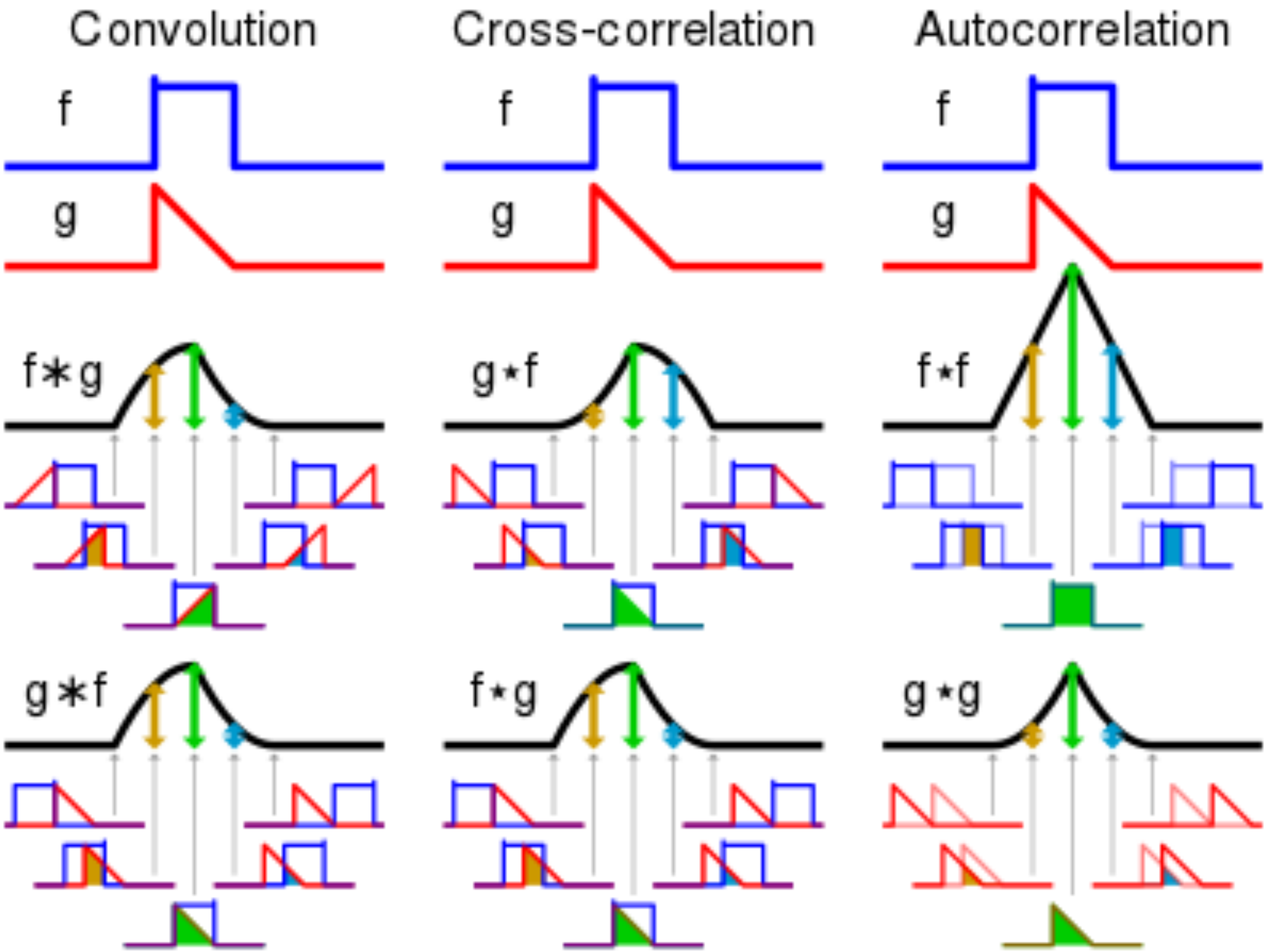
- Types of filters
  - Average – Smooth signal (normally it's used windows size 3/5) – eliminate white noise
  - Median Filter – Smooth signal – eliminate salt and pepper noise.
  - Gaussian Filter – Smooth signal – make more important central value - eliminate white noise
  - Hampel Filter – Eliminate outliers (if inside the window the value bigger than std dev, then median value is stored.

Savitzky–Golay Filter

- Savitzky–Golay Filter replace the value centered in the window with the polynomial regression of the values in the window.



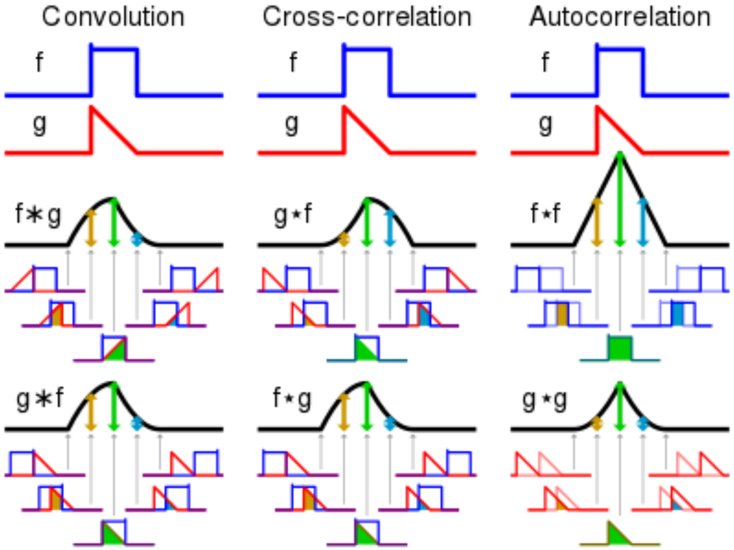
Correlation



https://en.wikipedia.org/wiki/Autocorrelation

Correlation

- Cross-correlation – Indicates how similar are the signals
- Auto-correlation – Indicates how synchronized are the signals.
- We can measure similarities between signal using correlation.
- We can detect if electrophysiological events are occurring simultaneously.



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Filtros

Imagen Original

Filtro Promedio

Filtro Gaussiano

Filtros

$$\frac{1}{9}\times\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

A

$$\frac{1}{16}\times\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix}$$

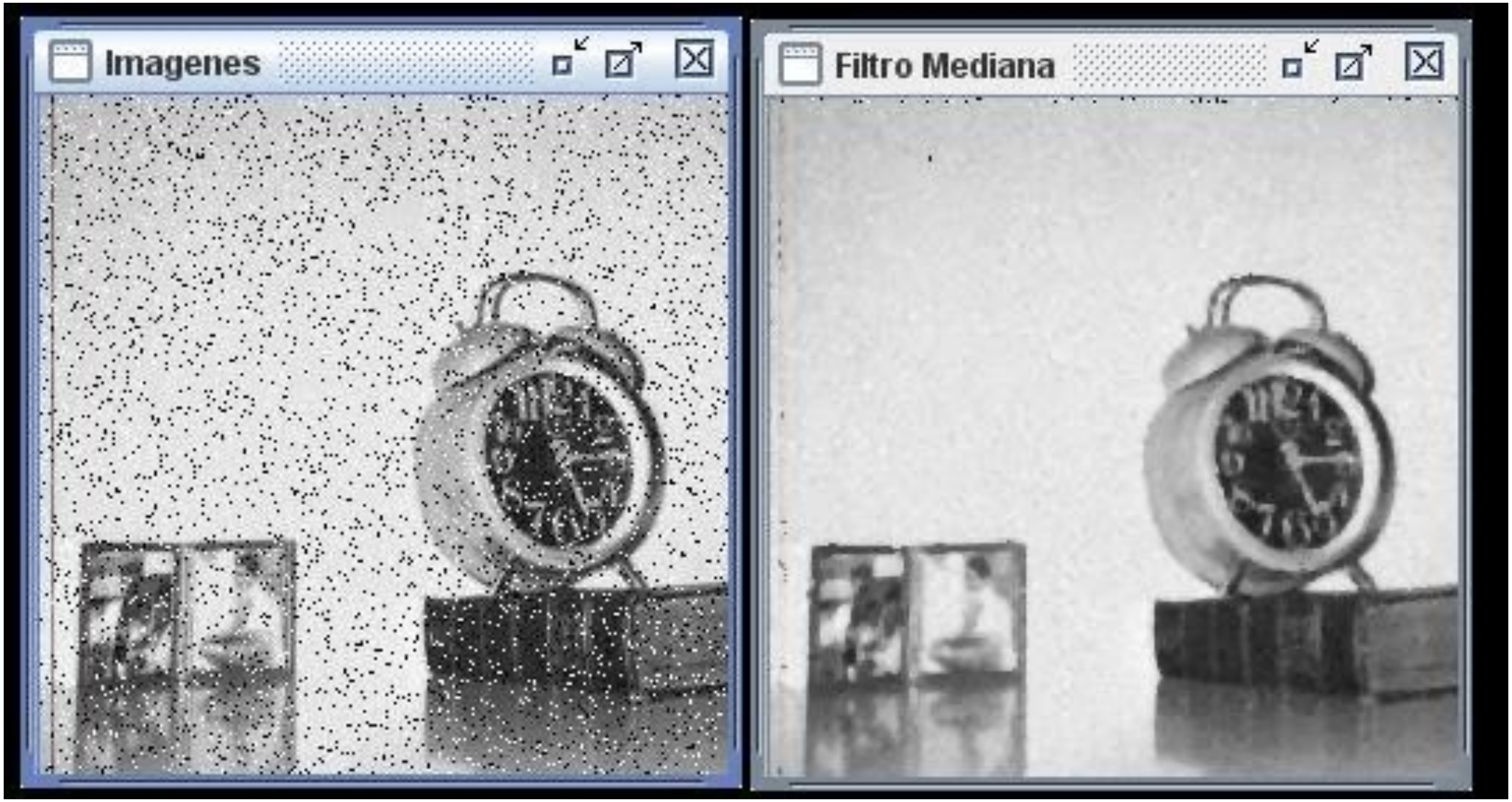
B

$$\frac{1}{10}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 1\end{bmatrix}$$

C

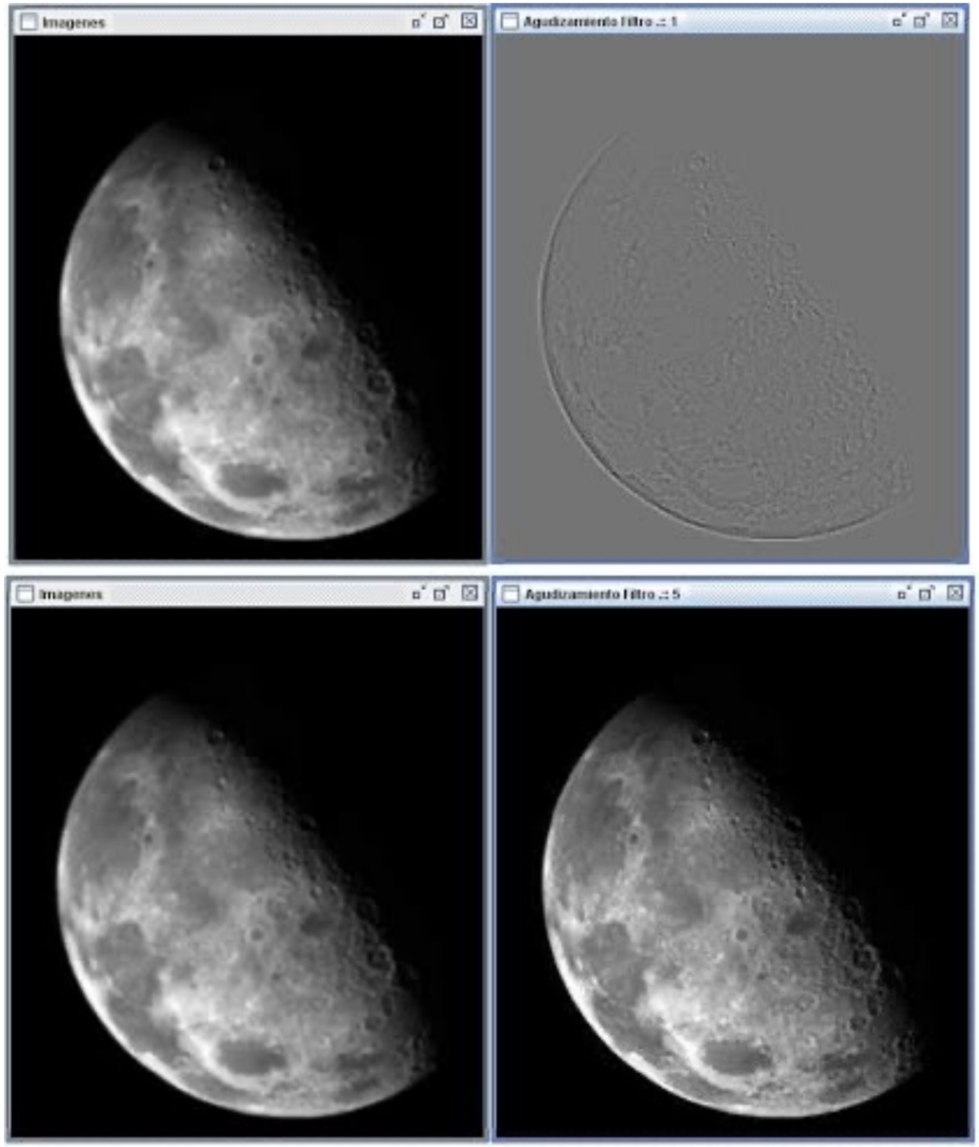
https://sites.google.com/site/lemared/

Filtros



https://sites.google.com/site/lemared/

Filtro



0	1	0
1	−4	1
0	1	0

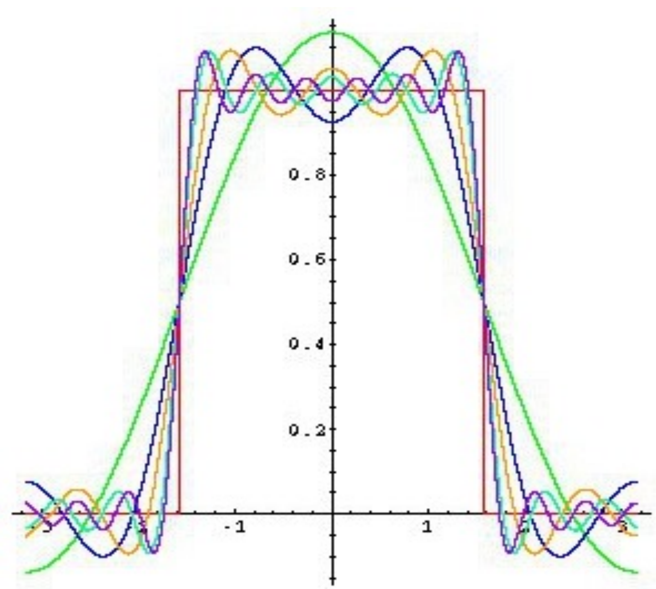
0	−1	0
−1	5	−1
0	−1	0

https://sites.google.com/site/lemared/

https://sites.google.com/site/lemared/

Serie de Fourier

A finales del siglo XVIII Jan Baptiste Joseph Fourier (1768-1830) descubrió un método que permite aproximar funciones periódicas mediante combinación lineal de funciones trigonométricas sencillas.



Serie de Fourier

Definición: Se llama serie de Fourier de una función f(x) en el intervalo [−L, L] a:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

Donde los coeficientes $\mathbf{a_0}$, $\mathbf{a_n}$ y $\mathbf{b_n}$ deben ser determinados.

Serie de Fourier

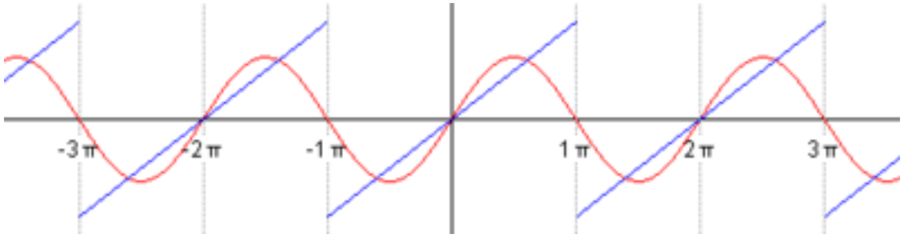
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

Los coeficientes $\mathbf{a_0}$, $\mathbf{a_n}$ y $\mathbf{b_n}$ están dados por:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \operatorname{sen} \frac{n\pi}{L} x \, \mathrm{d}x$$



Serie de Fourier

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x \qquad a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, \mathrm{d}x$$

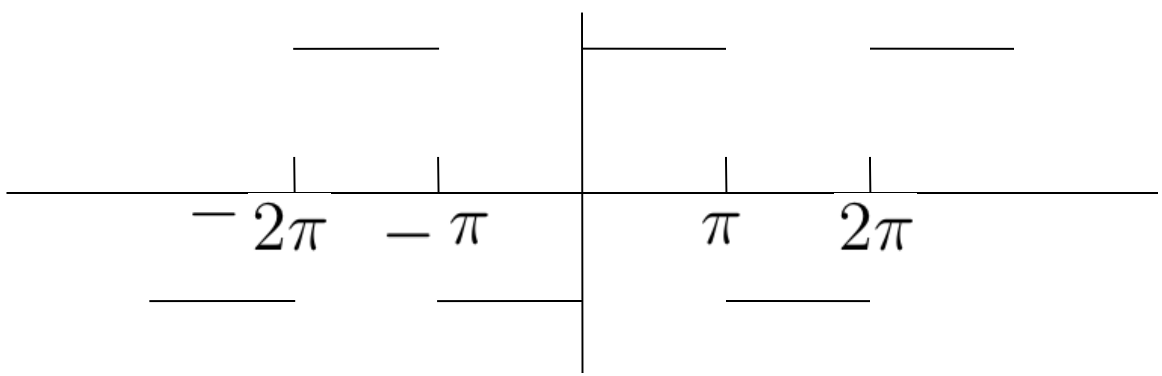
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \operatorname{sen} \frac{n\pi}{L} x \, \mathrm{d}x$$

$$\cos n\pi = (-1)^n \qquad \operatorname{sen} n\pi = 0$$

Serie de Fourier

Ejemplo: consideremos la función:

$$f(x) = \begin{cases} 1, & \text{si} \quad 0 \leq x \leq \pi; \\ -1, & \text{si} \quad \pi < x < 2\pi, \end{cases}$$



En este caso $2L = 2\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx$$

f(x) = -1 entre -п y 0   f(x) = 1 entre 0 y п

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx$$

$$a_n = \frac{1}{\pi}[-\int_{-\pi}^{0} f(x)cos(nx)dx + \int_{0}^{\pi} f(x)cos(nx)dx]$$

$$a_n = \frac{1}{\pi}[-\int_{-\pi}^{0} f(x)cos(nx)dx + \int_{0}^{\pi} f(x)cos(nx)dx]$$

$$\int cos(nx)dx = \frac{1}{n}sen(nx)$$

evaluada en $0, \pi$ ó $-\pi$ es igual a 0, por lo tanto:

$$a_n = 0$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sen(nx)dx$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

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$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[\frac{1}{n} - \frac{1}{n}cos(-n\pi) + \frac{1}{n} - \frac{1}{n}cos(n\pi)]$$

$$b_n = \frac{1}{\pi}[\frac{2}{n} - \frac{2}{n}cos(n\pi)]$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$-\int_{-\pi}^{0} sen(nx)dx = \frac{1}{n}cos(nx)|_{-\pi}^{0} = \frac{1}{n} - \frac{1}{n}cos(-n\pi)$$

$$\int_{0}^{\pi} sen(nx)dx = \frac{-1}{n}cos(nx)|_{0}^{\pi} = \frac{-1}{n}cos(n\pi) - \frac{-1}{n}$$

$$b_n = \frac{1}{\pi}[\frac{2}{n} - \frac{2}{n}cos(n\pi)]$$

$$b_n = \frac{2}{n\pi}[1 - cos(n\pi)]$$

$$cos(n\pi) = +1, \quad n \quad par$$

$$cos(n\pi) = -1, \quad n \quad impar$$

$$b_n = \frac{1}{\pi}[-\int_{-\pi}^{0} sen(nx)dx + \int_{0}^{\pi} sen(nx)dx]$$

$$b_n = \frac{2}{n\pi}[1 - cos(n\pi)]$$

$$cos(n\pi) = +1, \quad n \quad par$$

$$cos(n\pi) = -1, \quad n \quad impar$$

$$b_n = \frac{4}{n\pi}, \quad n \quad impar$$

$$a_0 = 0 \quad a_n = 0 \quad b_n = \frac{4}{n\pi}, \quad n \quad impar \quad 2L = 2\pi$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \operatorname{sen} \frac{n\pi}{L} x \right)$$

$$f(x) = \frac{4}{\pi} sen(x) + \frac{4}{3\pi} sen(3x) + \frac{4}{5\pi} sen(5x) + \cdots$$

$$f(x) = \frac{4}{\pi} \left[sen(x) + \frac{sen(3x)}{3} + \frac{sen(5x)}{5} + \cdots \right]$$

- Amplitud: $A_k = \sqrt{{a_k}^2 + {b_k}^2}$

- Fase: $\varphi_k = \tan^{-1} \frac{b_k}{a_k}$

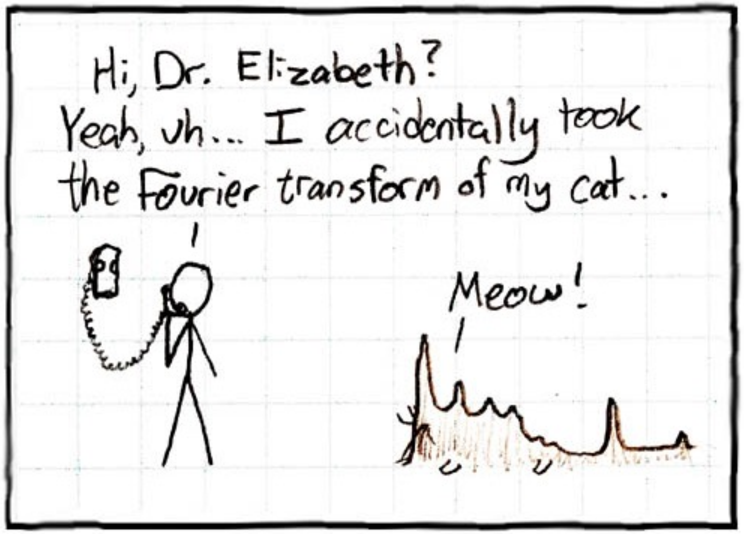
- Donde:

$$\bar{f}(x) = \sum_{k \in \mathbb{Z}_+} a_k \cos(k\omega_0 x) + b_k \sin(k\omega_0 x)$$

Fourier Transform

- Frequency domain transformations (Fourier)

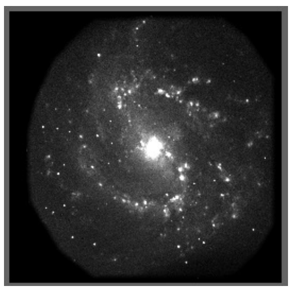
Jean-Baptiste Joseph Fourier (/ˈfʊəri.eɪ, -iər/;[1] French: [fuʁje]; 21 March 1768 – 16 May 1830) was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.[2]



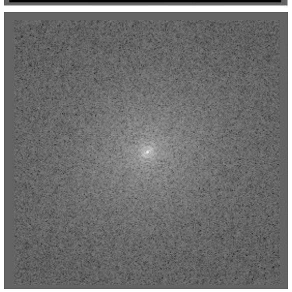
Joseph Fourier

Fourier Transform

Any periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies, called Fourier Series.



$$F(k,l) = \frac{1}{N^2} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} f(a,b)\, e^{-\iota 2\pi \left(\frac{ka}{N} + \frac{lb}{N}\right)}$$



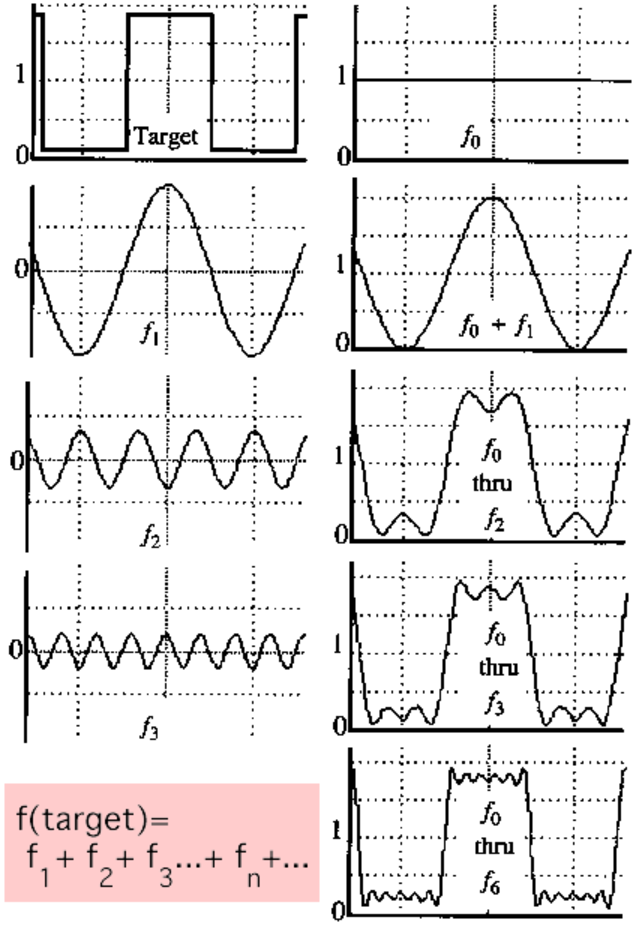
$$f(a,b) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k,l)\, e^{\iota 2\pi \left(\frac{ka}{N} + \frac{lb}{N}\right)}$$

Fourier Transform

- A sum of sinusoids
  - Building box

$$A\sin(\omega x + \phi)$$

- Assumptions
  - Periodic signals
  - More coefficients makes signal closer to the original.



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

Spatial domain ***f(x)*** → Fourier Transform → ***F(ω)*** Frequency domain

- For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine

$$A\sin(\omega x + \phi)$$

  – How can *F* hold both? Complex number trick!

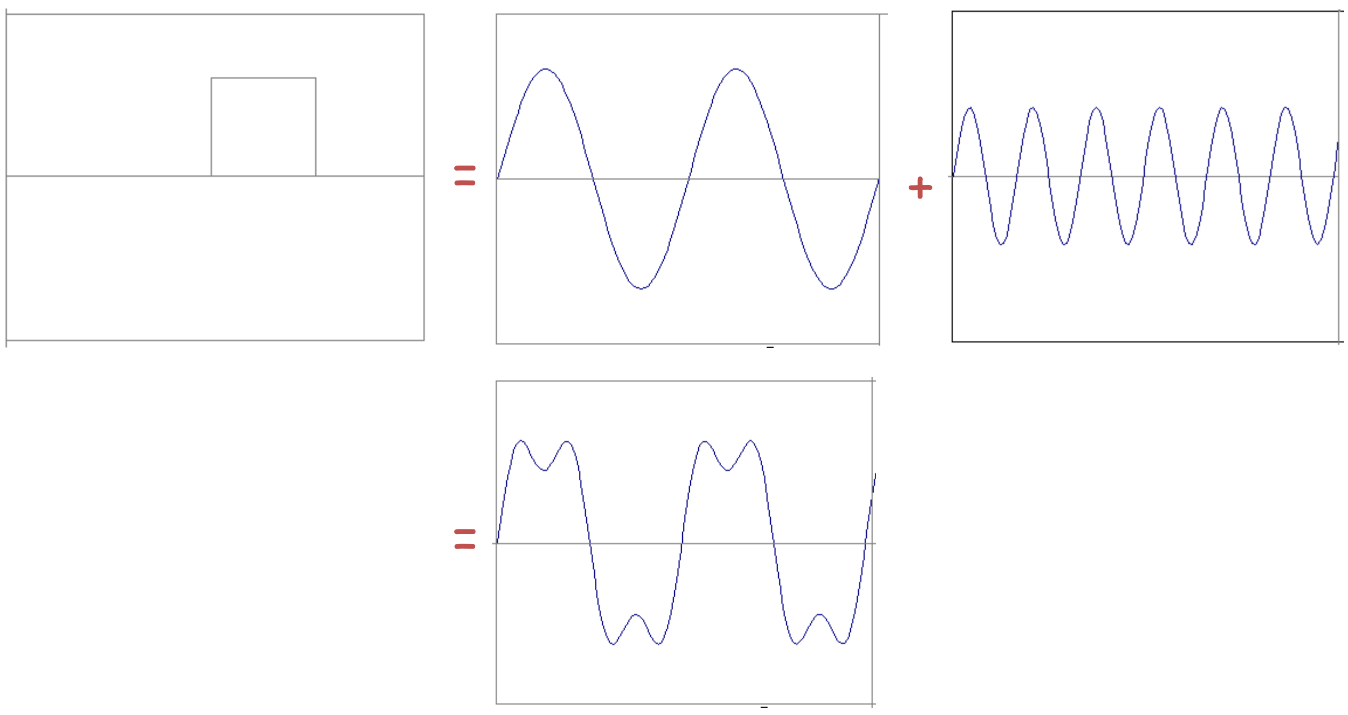
$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

F(ω) → Inverse Fourier Transform → ***f(x)***

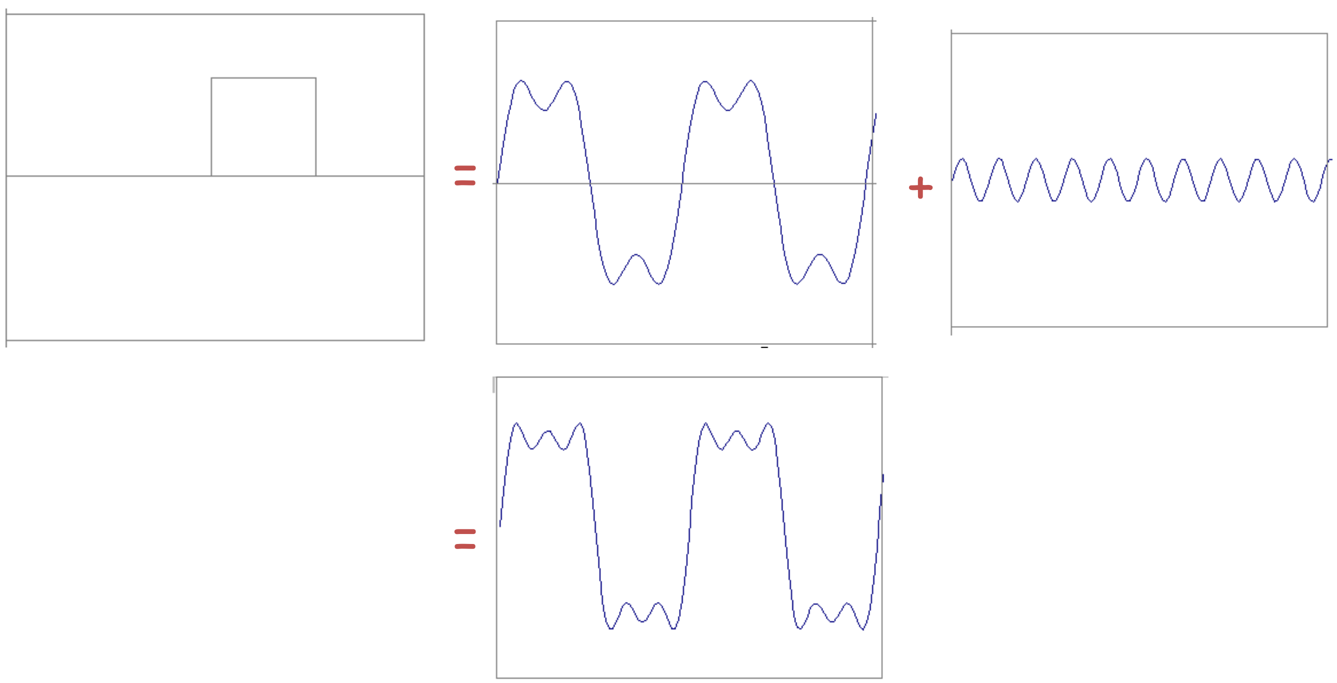
Fourier Transform

Example



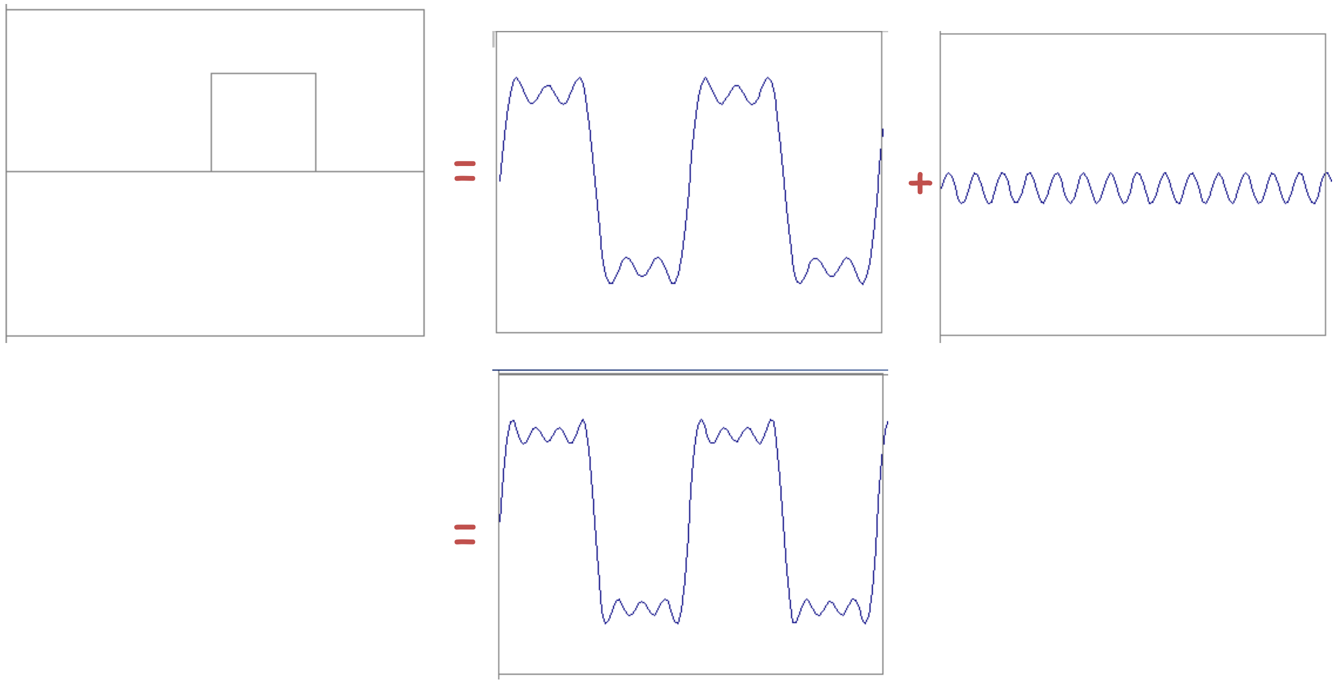
Fourier Transform

Example



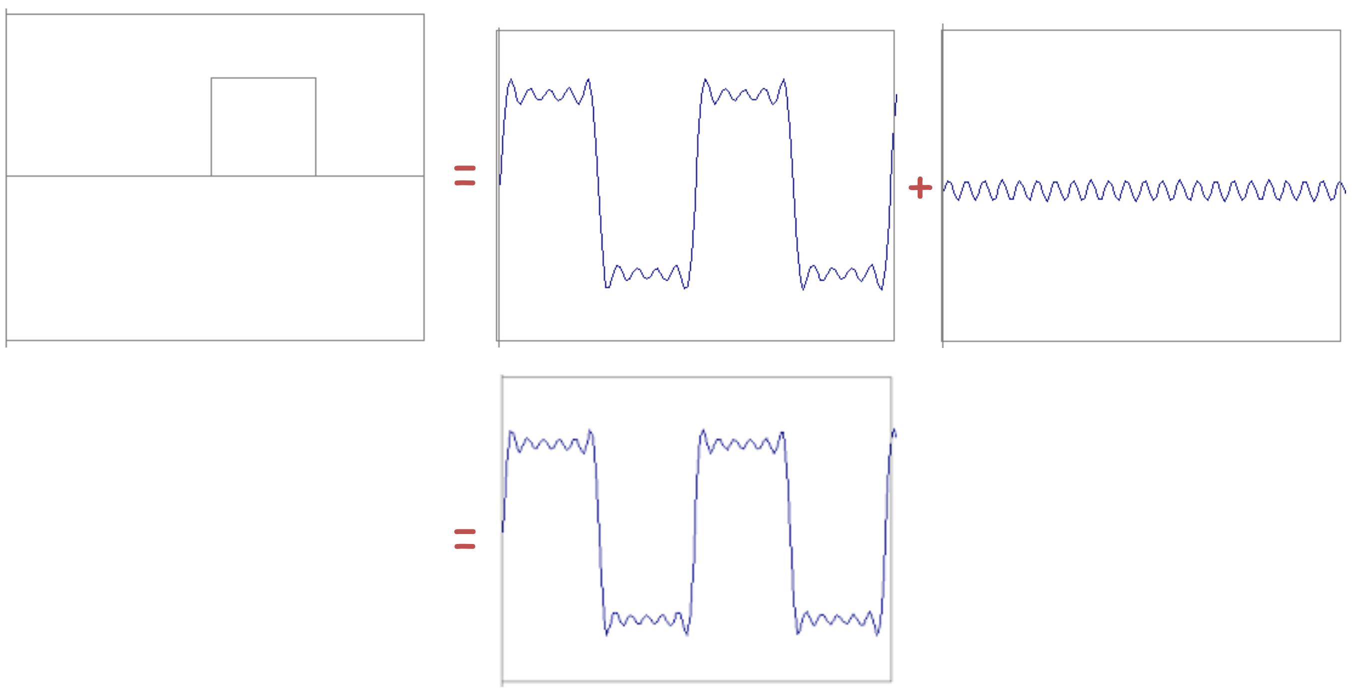
Fourier Transform

Example



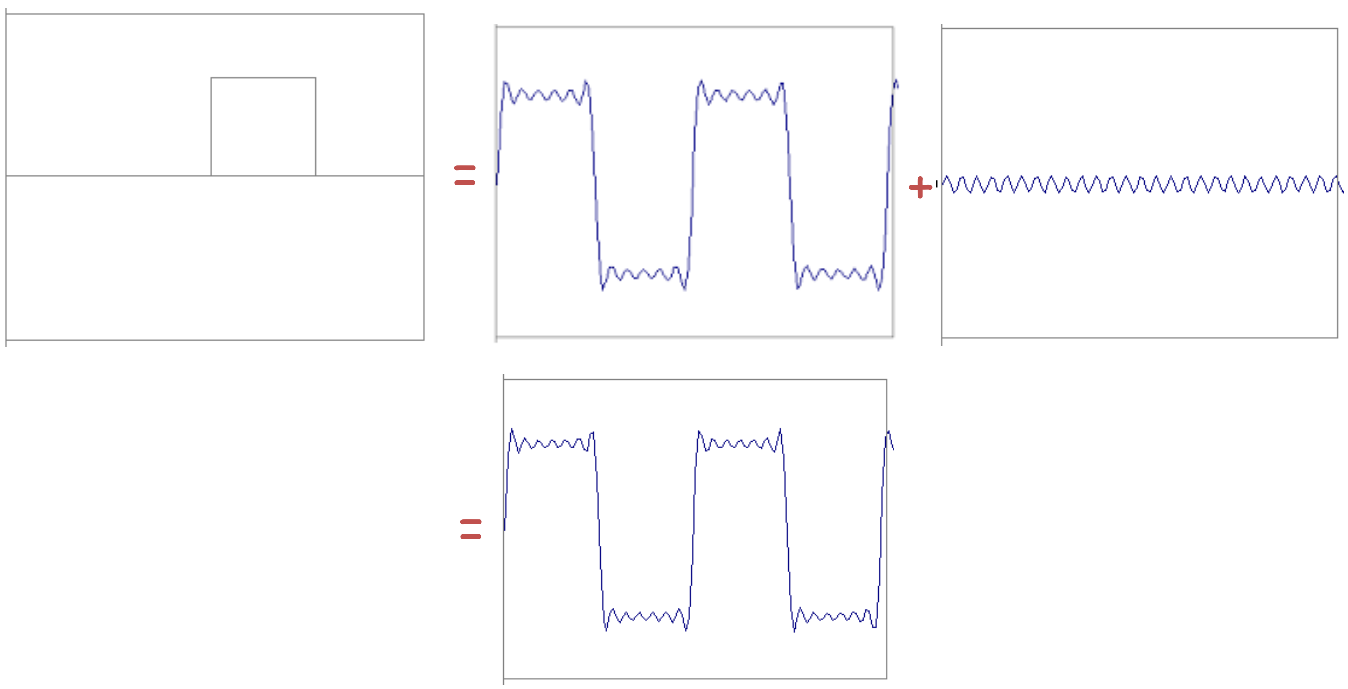
Fourier Transform

Example



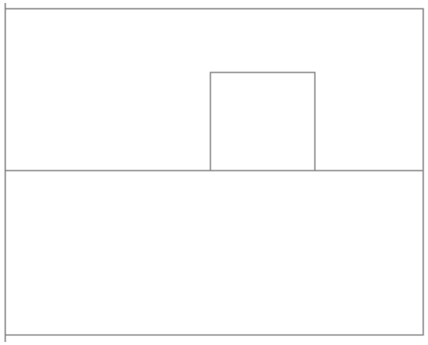
Fourier Transform

Example

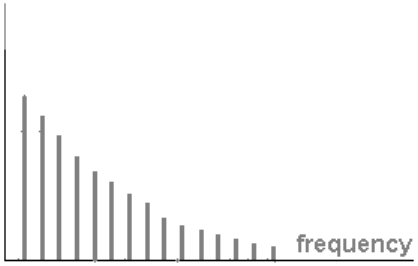


Fourier Transform

Example

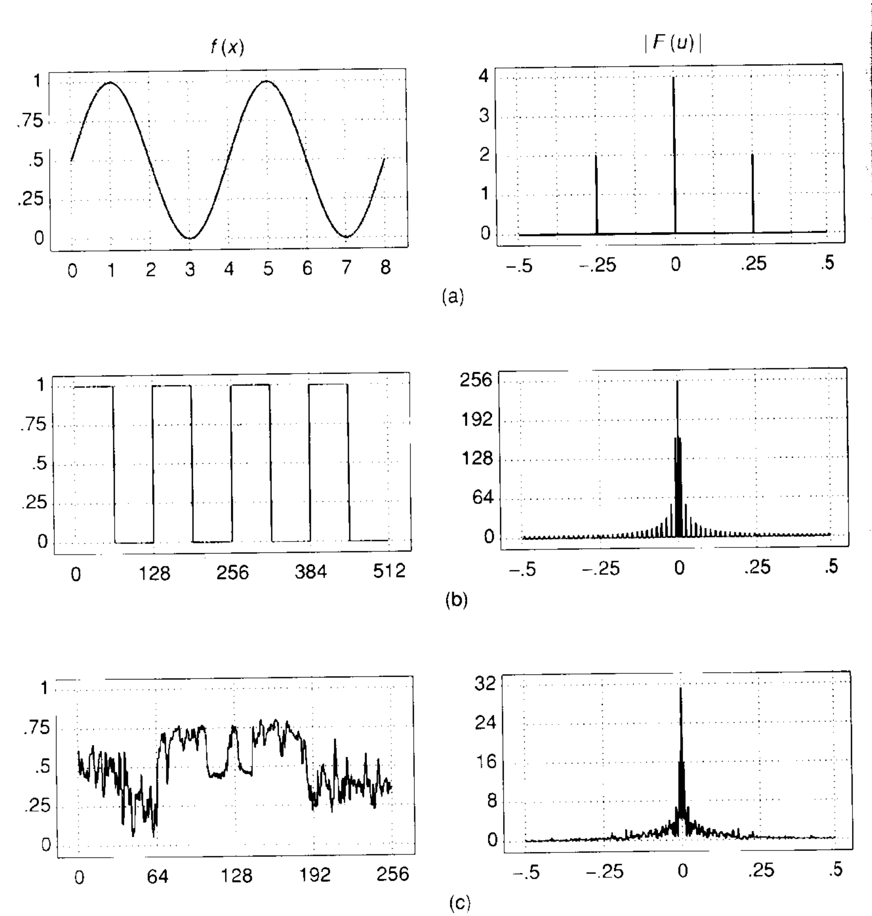


$$= A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$



Fourier Transform

Example

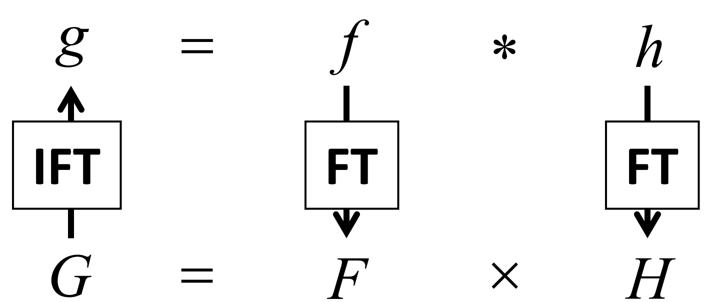


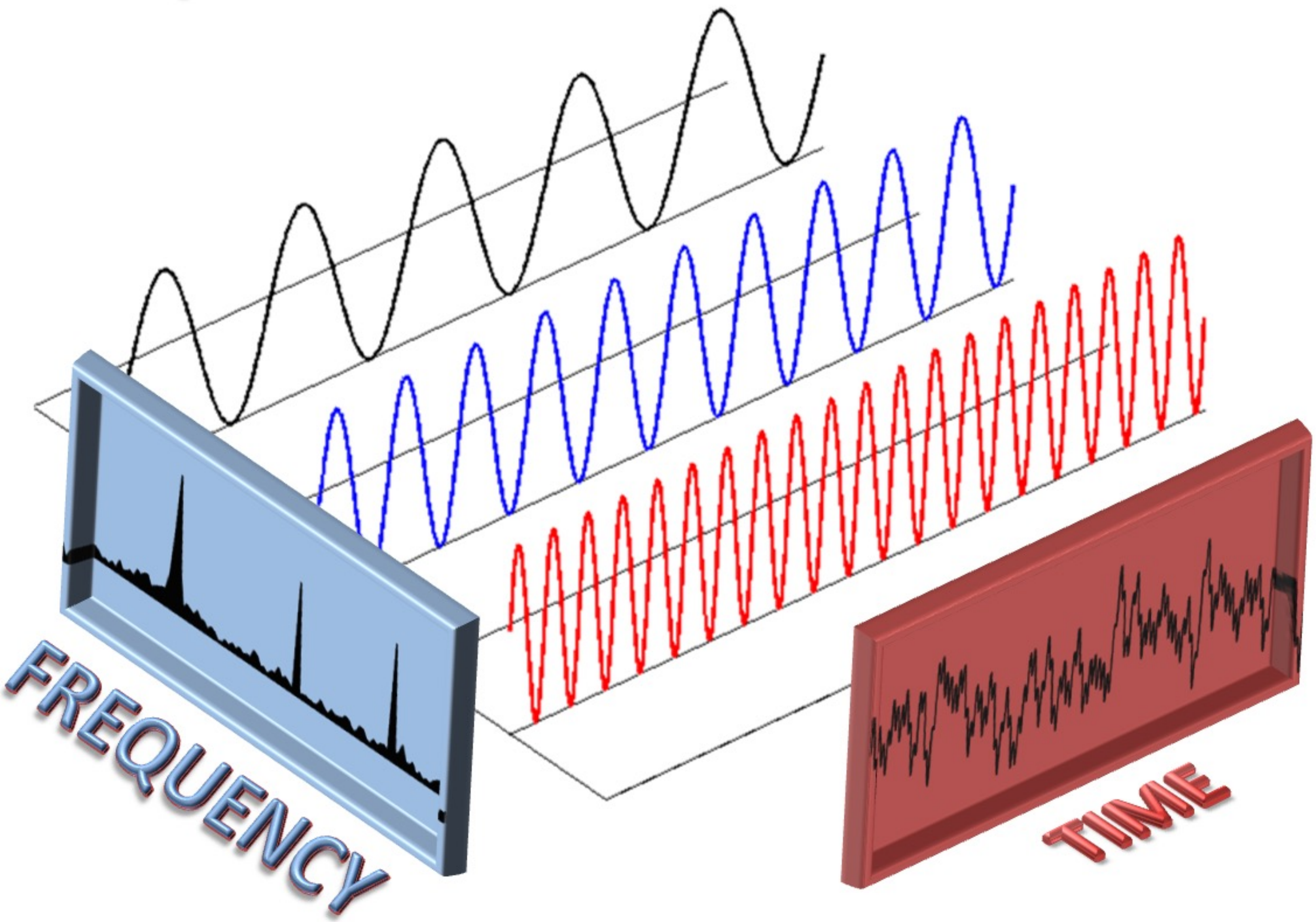
Fourier Transform

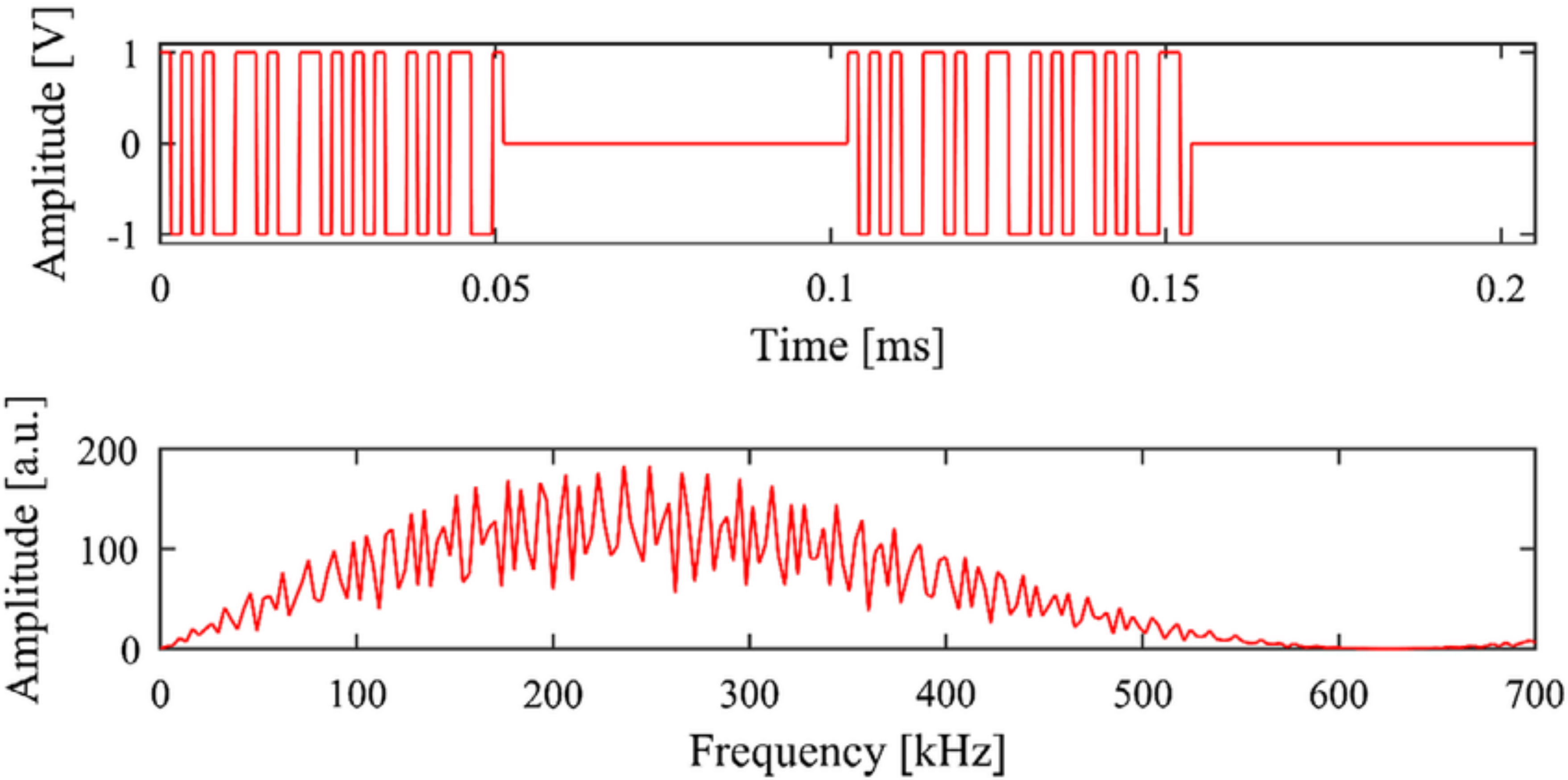
Fourier transform and convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

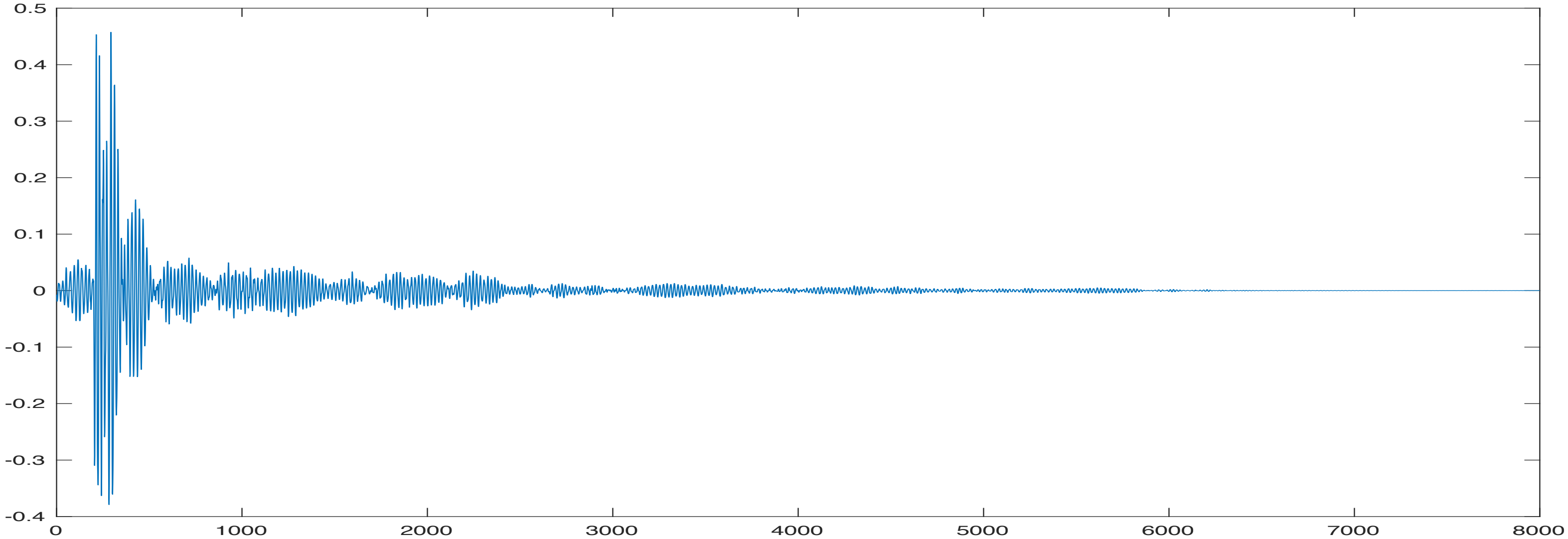
So, we can find *g(x)* by Fourier transform



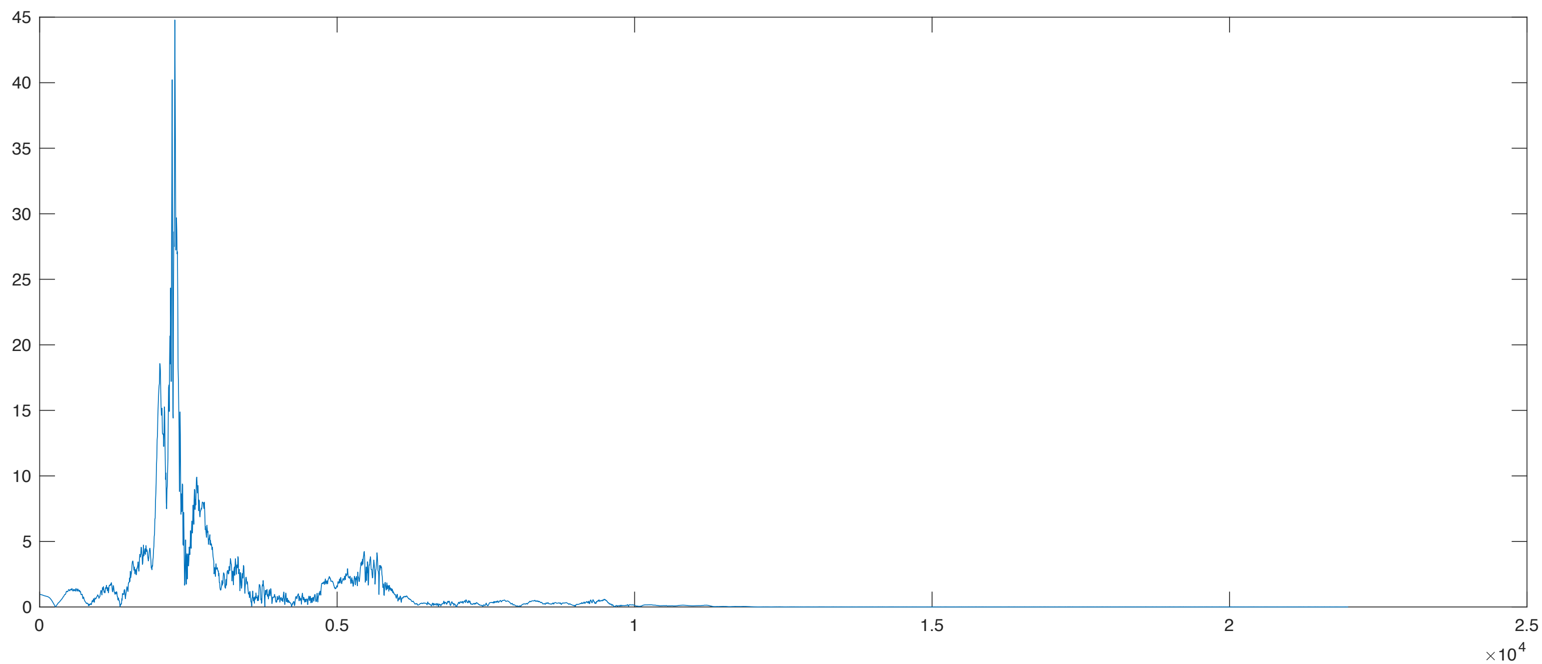

FREQUENCY
TIME


Amplitude [V]
1
0
-1
0
0.05
0.1
0.15
0.2
Time [ms]
Amplitude [a.u.]
200
100
0
0
100
200
300
400
500
600
700
Frequency [kHz]

Signal

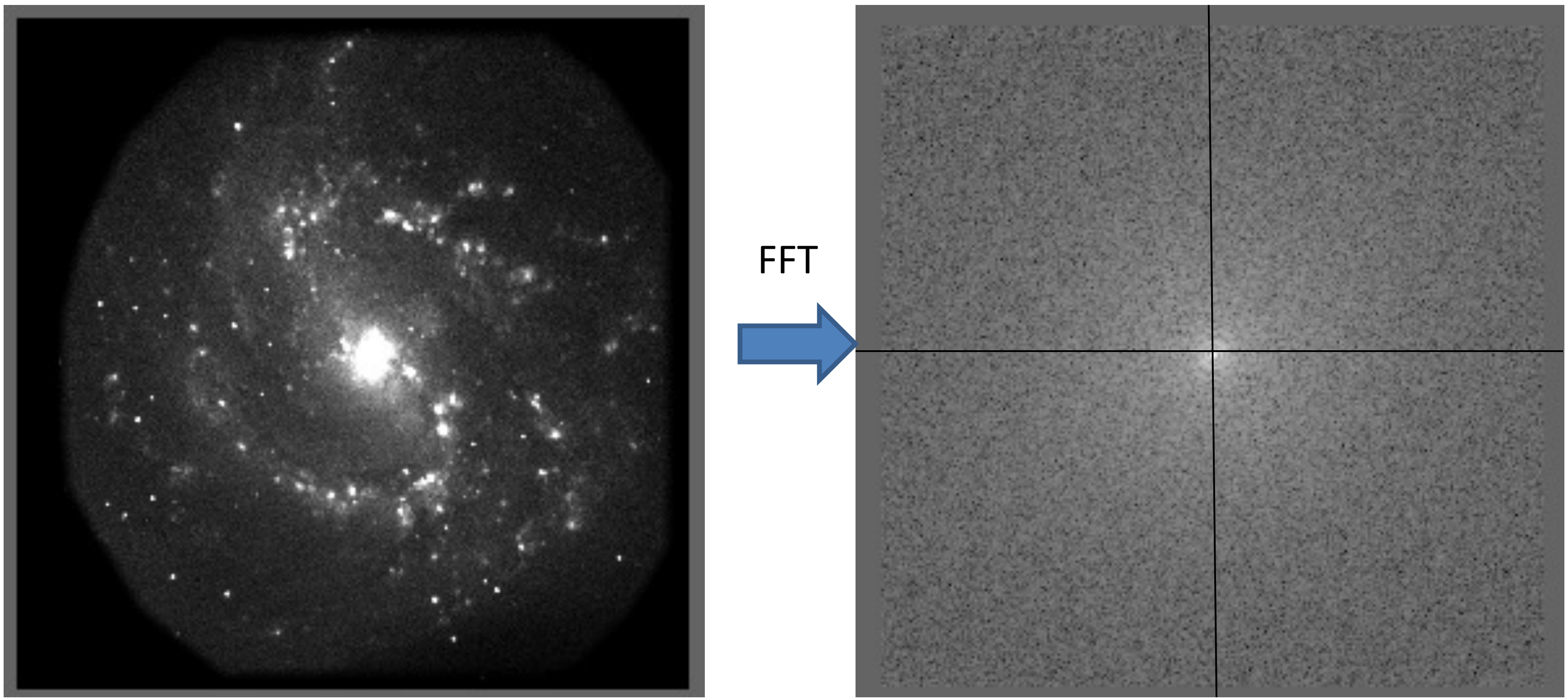


Frequency Spectrum

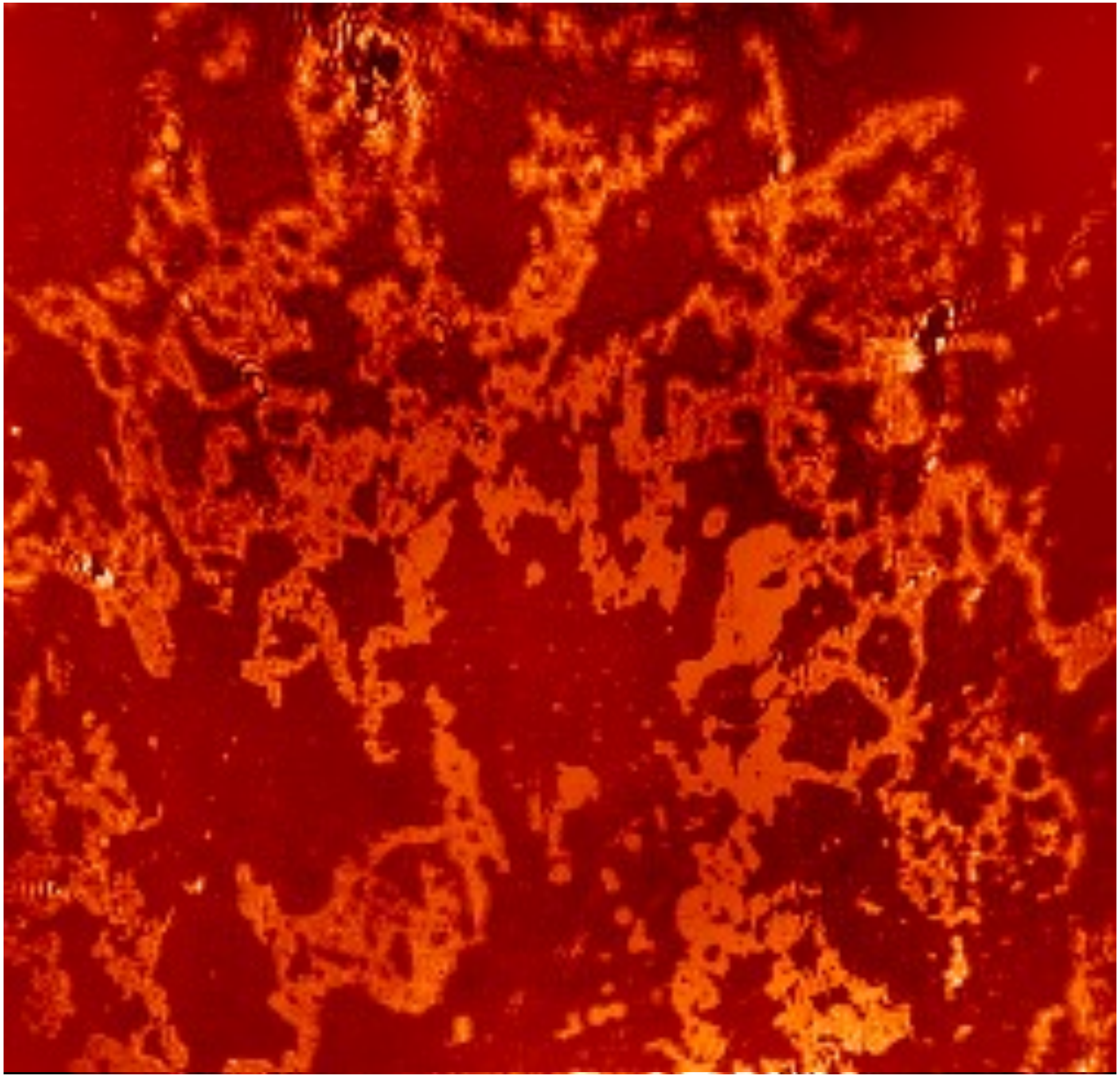


Fourier Transform

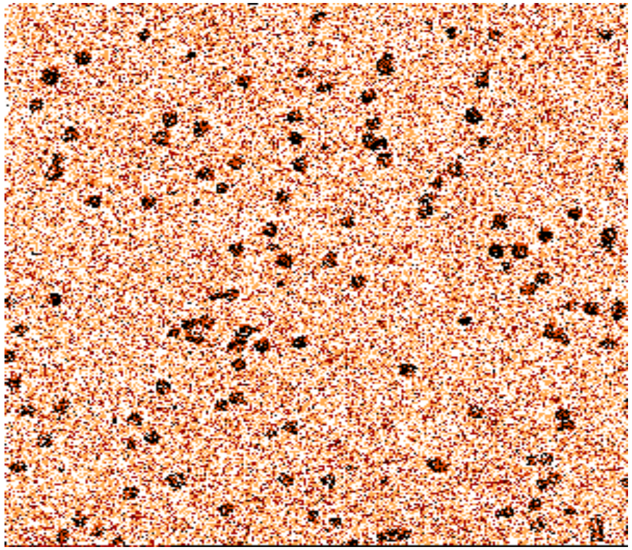
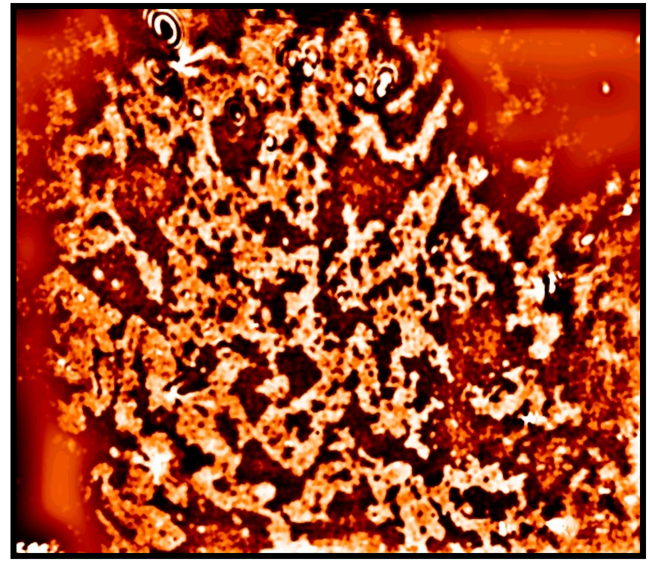
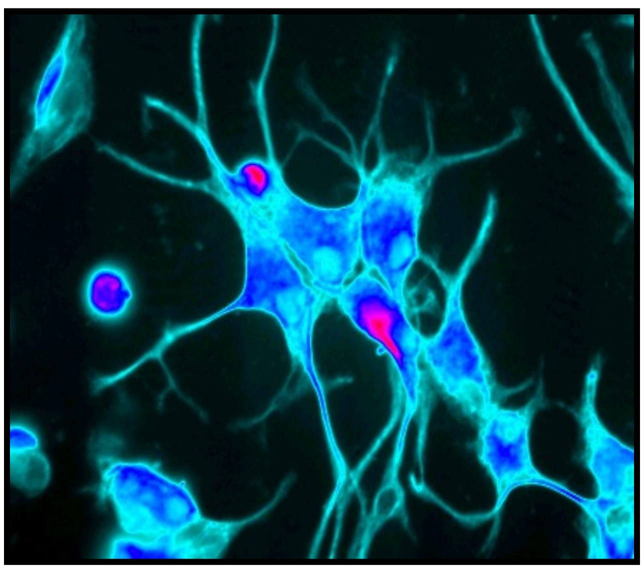
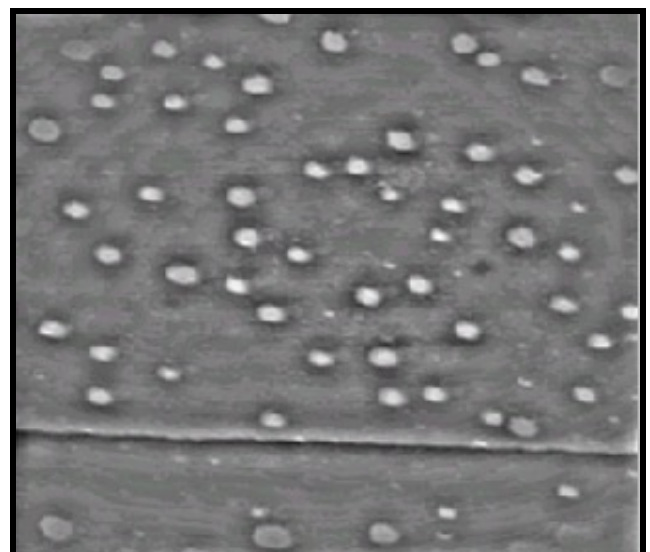
Fourier transform is symmetrical
x and y direction



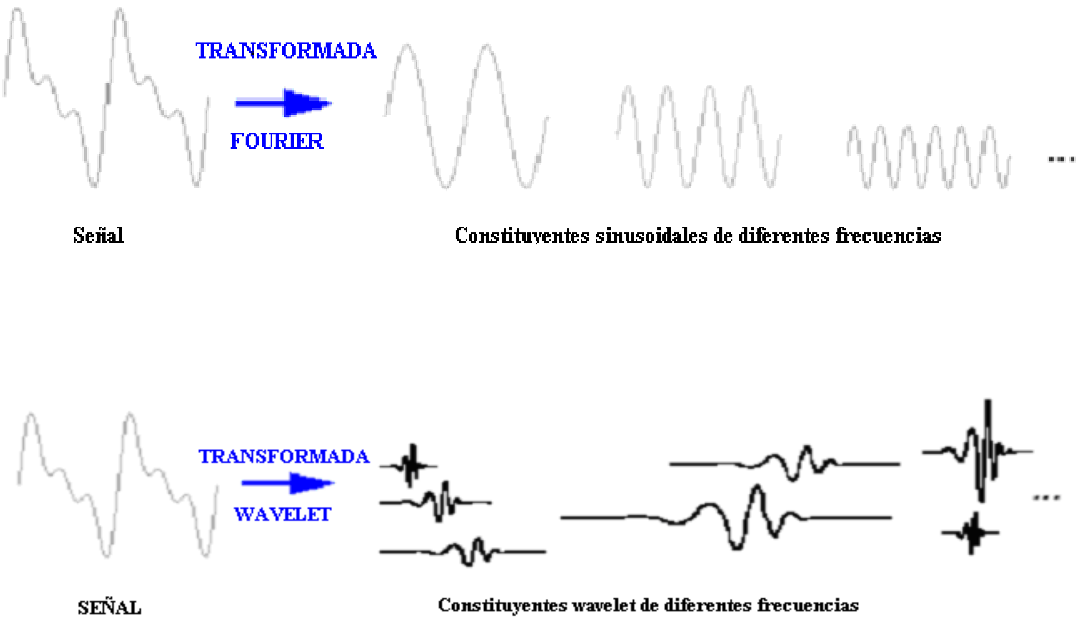
Fourier Transform



Fourier Transform

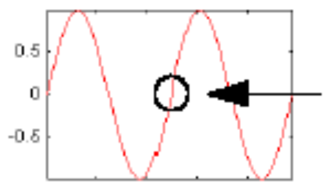





Wavelets Transform

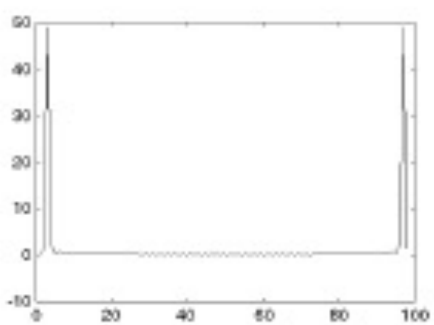


Wavelets Transform

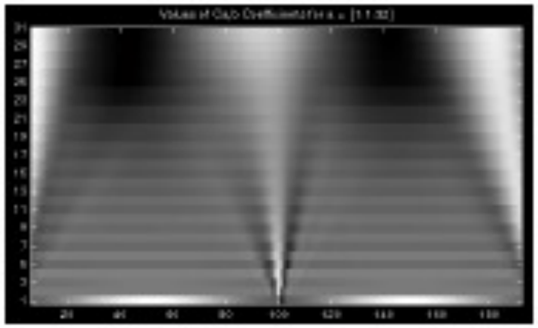
- Imagine a sinusoidal signal with a small discontinuity:



- **Fourier does not see the discontinuity.**
- **Wavelet shows exactly the location of the discontinuity in time.**



Fourier coeficients



Wavelet coeficients

Wavelets Transform

- Matematically, Fourier analysis representsed by the Fourier transform divide the original signal in a sum of sinusoidal signals.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt,$$

- **Wavelets transform is defined as a sum all time of the signal multiplied by a scale, changing the wavelet function. The wavelet coefficient result are then in terms of scale and position.**

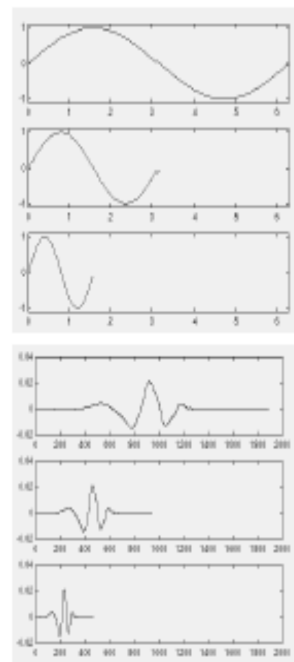
$$C(scale, position) = \int_{-\infty}^{\infty} f(t) \psi(scale, position, t) dt$$

Wavelets Transform

- **<u>Scaling of wavelet</u>**

  - **Scale a wavelet means shrink or elongate is denominated scale factor**

  - **In sinusoidal the scale factor is easy to see….**



$f(t) = \text{Seno}(t)\ ;\ \ a = 1$

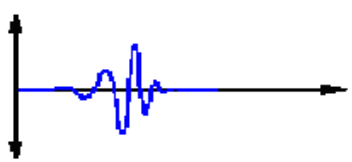
$f(t) = \text{Seno}(2t)\ ;\ \ a = 1/2$

$f(t) = \text{Seno}(4t)\ ;\ \ a = 1/4$

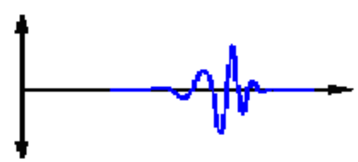
$f(t) = \Psi(t)\ ;\ \ a = 1$

$f(t) = \Psi(2t)\ ;\ \ a = 1/2$

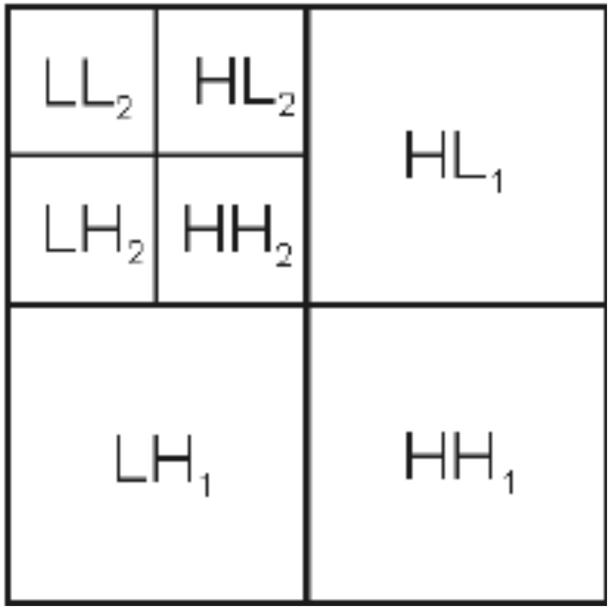
$f(t) = \Psi(4t)\ ;\ \ a = 1/4$

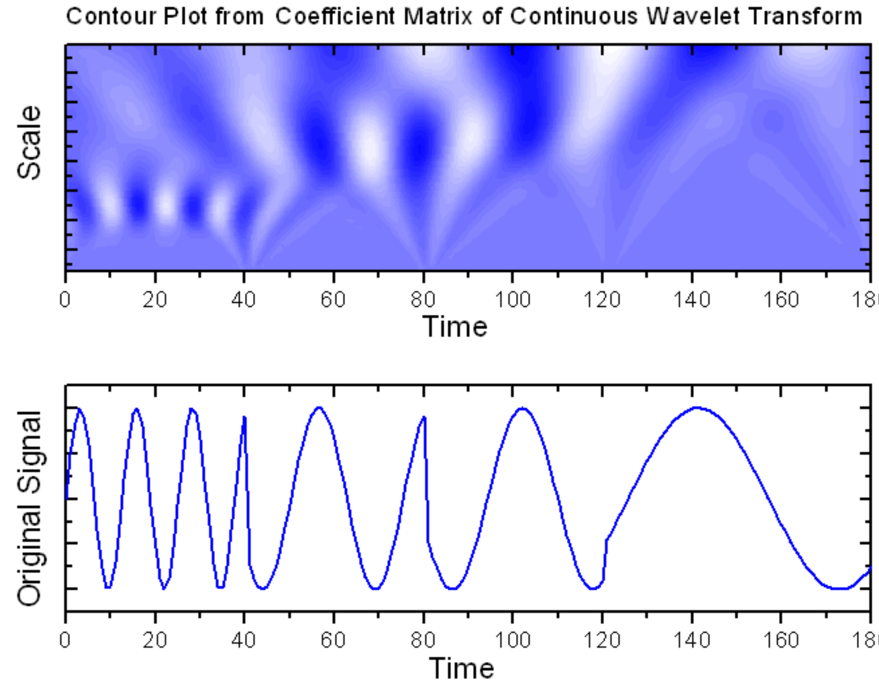


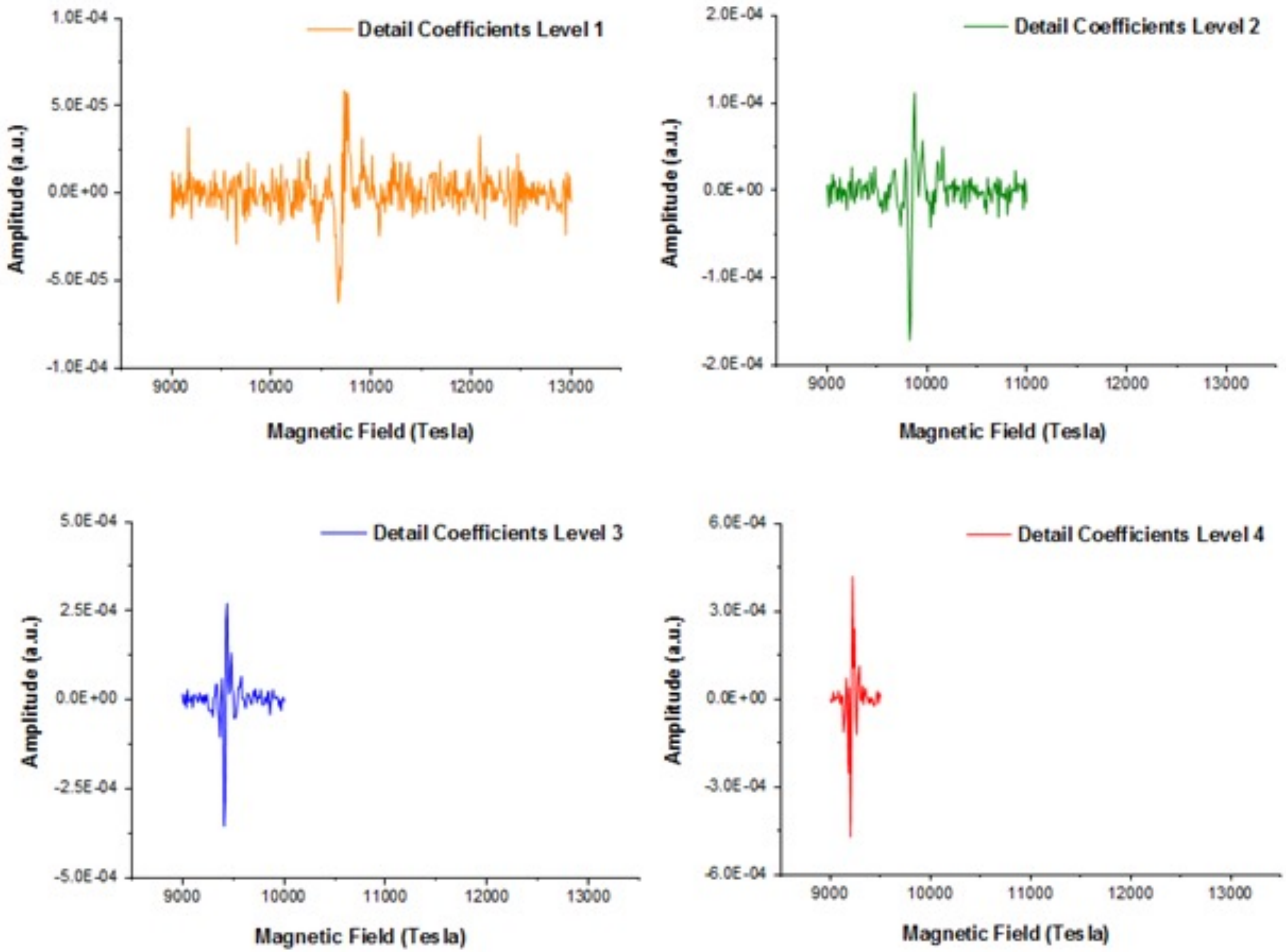
Función Wavelet $\Psi(t)$



Función Wavelet desplazada $\Psi(t - k)$


LL2
HL2
LH2
HH2
HL1
LH1
HH1


Contour Plot from Coefficient Matrix of Continuous Wavelet Transform
Scale
0 20 40 60 80 100 120 140 160 180
Time
Original Signal
0 20 40 60 80 100 120 140 160 180
Time



Detail Coefficients Level 1
1.0E-04
5.0E-05
0.0E+00
-5.0E-05
-1.0E-04
Amplitude (a.u.)
9000
10000
11000
12000
13000
Magnetic Field (Tesla)
Detail Coefficients Level 2
2.0E-04
1.0E-04
0.0E+00
-1.0E-04
-2.0E-04
Amplitude (a.u.)
9000
10000
11000
12000
13000
Magnetic Field (Tesla)
Detail Coefficients Level 3
5.0E-04
2.5E-04
0.0E+00
-2.5E-04
-5.0E-04
Amplitude (a.u.)
9000
10000
11000
12000
13000
Magnetic Field (Tesla)
Detail Coefficients Level 4
6.0E-04
3.0E-04
0.0E+00
-3.0E-04
-6.0E-04
Amplitude (a.u.)
9000
10000
11000
12000
13000
Magnetic Field (Tesla)

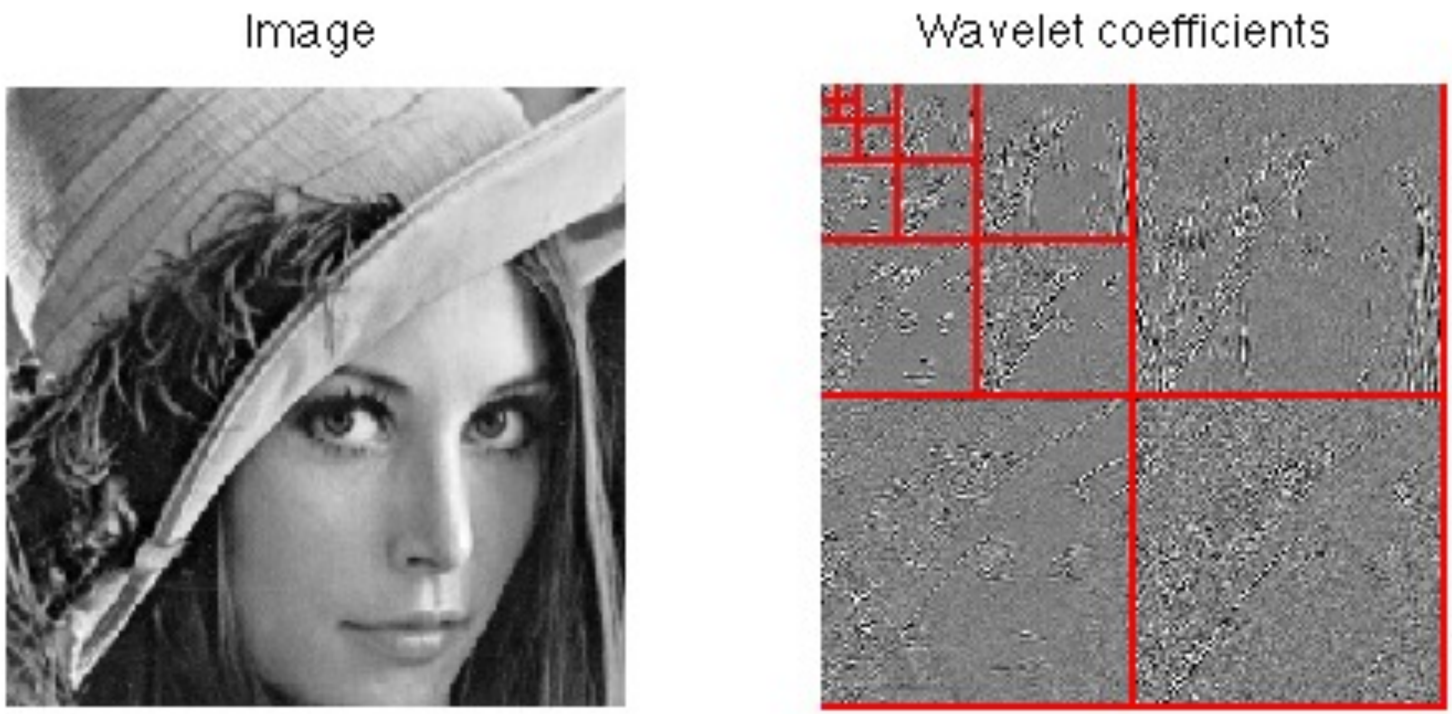
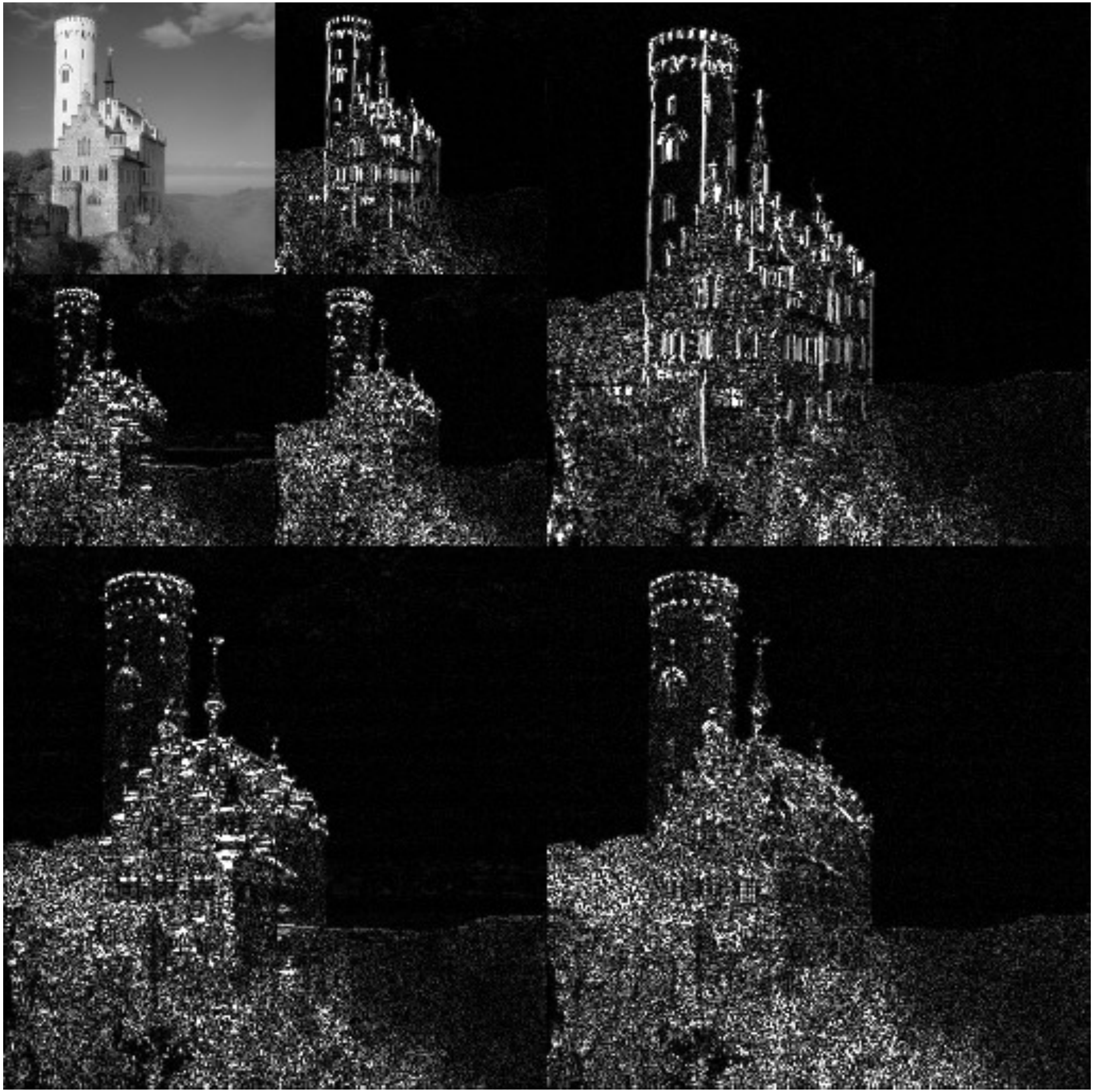


Image
Wavelet coefficients



Noisy Image

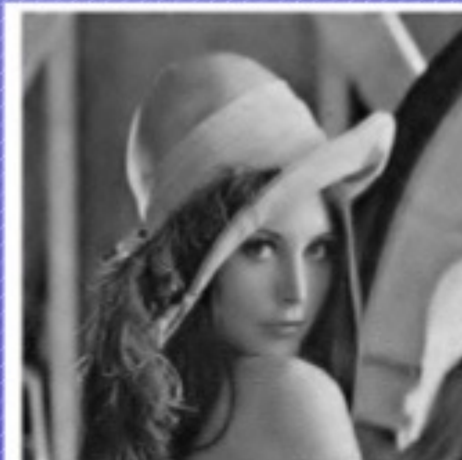
Filtered Image

FILTRADO EN ESPACIO DE FOURIER:

Se eliminan las frecuencias más altas

FILTRADO EN ESPACIO DE WAVELETS:

Se eliminan los coeficientes menores.



With Trans. Ave.

Questions?